A LINEAR TRANSFORMATION WHOSE VARIABLES AND COEFFICIENTS ARE SETS OF POINTS

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Introduction. While the theory of the linear transformation has been developed in great detail, attention has seldom[1] been called to the transformation T in which variables and coefficients are sets of points. Doubtless the nonexistence of a unique inverse transformation has occasioned this neglect. In this paper the writer studies the iteration of T.

Consider first the transformation

$$T:\quad \begin{aligned} x_1 &= a_{11}x_1' + a_{12}x_2' \\ x_2 &= a_{21}x_1' + a_{22}x_2' \end{aligned},$$

whose *set matrix* is

$$M = \left\| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right\|,$$

where the a's and x's are sets of points, and the indicated sums and products refer to set operations. Applying T to the primed variables, we have the product transformation

$$T^2:\quad \begin{aligned} x_1 &= \overset{(2)}{a_{11}}\, x_1'' + \overset{(2)}{a_{12}}\, x_2'' \\ x_2 &= \overset{(2)}{a_{21}}\, x_1'' + \overset{(2)}{a_{22}}\, x_2'' \end{aligned},$$

of set matrix

$$M^2 = \left\| \begin{matrix} \overset{(2)}{a_{11}} & \overset{(2)}{a_{12}} \\ \overset{(2)}{a_{21}} & \overset{(2)}{a_{22}} \end{matrix} \right\|,$$

where

$$\begin{aligned} &\overset{(2)}{a_{11}} = a_{11} + a_{12}a_{21}, && \overset{(2)}{a_{12}} = a_{11}a_{12} + a_{12}a_{22}, \\ &\overset{(2)}{a_{21}} = a_{21}a_{11} + a_{22}a_{21}, && \overset{(2)}{a_{22}} = a_{21}a_{12} + a_{22}. \end{aligned} \tag{1}$$

Transforming in turn each new set of variables, we obtain product transformations $T^3, T^4, \cdots$, whose set matrices are $M^3, M^4, \cdots$.

Presented to the Society, December 30, 1941 under the title *On powers of a matrix whose elements are sets of points;* received by the editors August 25, 1941.

[1] Lowenheim, *Über Transformationen im Gebietekalkül*, Mathematische Annalen, vol. 73 (1913), pp. 245–272; *Gebietsdetermination*, Mathematische Annalen, vol.79 (1919), pp. 223–236.

Set matrices $M(a_{ij})$ and $M'(a'_{ij})$ are defined to be *equal* if $a_{ij}=a'_{ij}$, $i, j=1, 2, \cdots, n$, while if $a_{ij}\subset a'_{ij}$,[2] then we shall write $M\subset M'$. A sequence M_k, $k=1, 2, \cdots$, of set matrices is *increasing* or *decreasing* according as $M_k\subset M_{k+1}$, or $M_{k+1}\subset M_k$.

From (1) follow $a_{11}\subset a_{11}^{(2)}$ and $a_{22}\subset a_{22}^{(2)}$. The assumptions $a_{12}\subset a_{12}^{(2)}$ and $a_{21}\subset a_{21}^{(2)}$ are equivalent to

$$a_{12} + a_{21} \subset a_{11} + a_{22}, \tag{2}$$

which is a necessary and sufficient condition that $M\subset M^2$. But from (1) we have $a_{12}^{(2)}+a_{21}^{(2)}\subset a_{11}^{(2)}+a_{22}^{(2)}$, whence by (2) the inclusion $M^2\subset M^4$, and the following theorem is established.

THEOREM 1. *The sequences M^{2k} and M^{2k+1}, $k=1, 2, \cdots$, of second order set matrices are increasing.*

From (1) follow also $a_{12}^{(2)}\subset a_{12}$ and $a_{21}^{(2)}\subset a_{21}$, while the assumptions $a_{11}^{(2)}\subset a_{11}$ and $a_{22}^{(2)}\subset a_{22}$ are equivalent to

$$a_{12}a_{21} \subset a_{11}a_{22}, \tag{3}$$

a necessary and sufficient condition that $M^2\subset M$. From (1) comes $a_{12}^{(2)}a_{21}^{(2)}\subset a_{11}^{(2)}a_{22}^{(2)}$, whence by (3) the inclusion $M^4\subset M^2$, and the following theorem.

THEOREM 2. *The sequences M^{2k} and M^{2k+1}, $k=1, 2, \cdots$, of second order set matrices are decreasing.*

This theorem follows at once.

THEOREM 3. *Even powers of a second order transformation are identical; likewise odd powers beyond the first.*

The general case. Consider the transformation

$$T: \quad x_i = \sum_{j=1}^{n} a_{ij}x'_j, \qquad i = 1, 2, \cdots, n,$$

of set matrix

$$M = \left\| \begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right\|,$$

where the variables and coefficients are sets of points in a space whose generality is limited only by the following assumption.

[2] The symbol $\subset$ denotes inclusion.

(A) *The elements a_{ij} of M are independent point sets.* That is, if products $P_1, P_2, \cdots, P_k, \cdots$ of elements of M are such that

$$P_1 \subset \sum_{k \neq 1} P_k,$$

there is a subscript $k = k'$ such that[3] $P_1 \subset P_{k'}$.

The following theorem will presently be established.

THEOREM 4. *The iteration of the transformation T of order n leads to at most $(n-1)^2+N$ distinct transformations, where N is the least common multiple of $1, 2, \cdots, n$, and at most N of these transformations recur periodically. If the coefficients of T are independent, there are precisely $(n-1)^2+N$ distinct transformations of which N recur periodically.*

We shall denote by $a_{ij}^{(p)}$ the element in the ith row and jth column of M^p. Clearly $a_{ij}^{(p)}$ is a sum of products of the form

$$P = a_{ik_2}a_{k_2k_3}a_{k_3k_4} \cdots a_{k_pj},$$

for all distinct ways of selecting the subscripts $k_2, k_3, \cdots, k_p$, from the integers $1, 2, \cdots, n$. The characteristic interlocking form k_rk_s, k_sk_t of the subscripts of consecutive factors of P will be expressed in the word *proper*. Thus $a_{13}a_{32}a_{23}a_{32}a_{22}$ is a term of $a_{12}^{(5)}$ and a proper product, as opposed to the identical point set $a_{13}a_{32}a_{23}a_{22}$.

The *order* of a product P is the number of factors occurring in the product. The orders of products $P, C, \cdots$, will be denoted by $p, c, \cdots$. A *cycle* of P is a proper product $a_{k_1k_2}a_{k_2k_3} \cdots a_{k_ck_1}$ of factors of P in which the subscripts $k_1, k_2, \cdots, k_c$ are distinct. It is convenient to denote such a cycle by C_{k_1}. Thus $a_{12}a_{23}a_{33}a_{32}$ involves the cycles $C_2 = a_{23}a_{32}$, $C_3 = a_{33}$, and $C_3' = a_{32}a_{23}$. A *closed cycle* of P is one whose factors occur consecutively in P. A product P is *c-cyclic* if every product of c consecutive factors is a closed cycle. It follows that these closed cycles are cyclic permutations of a single cycle C, known as the defining cycle of the product P. Thus $a_{23}a_{31}a_{12}a_{23}a_{31}$ is 3-cyclic, with the defining cycle $C_2 = a_{23}a_{31}a_{12}$.

Proper products are *coterminal* if corresponding terminal subscripts are equal. Thus $P = a_{12}a_{23}a_{34}$ and $P_1 = a_{15}a_{54}$ are coterminal. Such products clearly are terms of elements similarly situated in set matrices M^p and $M^{p'}$. It is convenient to denote a proper product $a_{ik_2}a_{k_2k_3} \cdots a_{k_pj}$ by P_{ij}. If coterminal proper products P_{ij}, P_{ij}' are

[3] Thus, under (A), such inclusions as $a_{12}a_{23} \subset P = a_{14} + a_{13}a_{32}$ cannot hold, since P does not involve one of the terms $a_{12}, a_{23}, a_{12}a_{23}$.

such that $p'_{ij} < p_{ij}$ while $P_{ij} \subset P'_{ij}$, we say that P'_{ij} is a *contraction* of P_{ij}, while P_{ij} is an *expansion* of P'_{ij}. Thus $a_{12}a_{23}$ is a contraction of $a_{12}a_{24}a_{42}a_{23}$, while the latter product is an expansion of the former.

The following lemmas are immediate consequences of the preceding definitions.

LEMMA 1. *A proper product of order exceeding $n-1$ involves a closed cycle.*

LEMMA 2. *The deletion of a closed cycle from a proper product of greater order yields a contraction of the product.*

LEMMA 3. *The insertion of a cycle C_{k_s} into a proper product immediately following a factor $a_{k_r k_s}$, or immediately preceding a factor $a_{k_s k_t}$, yields an expansion of the product. Thus the appropriate insertion of $C_1 = a_{14}a_{41}$ into $P_{12} = a_{13}a_{31}a_{12}$ yields $P'_{12} = a_{13}a_{31}a_{14}a_{41}a_{12}$, or $a_{14}a_{41}a_{13}a_{31}a_{12}$.*

Consider the sequence

$$P_{ij},\ P'_{ij},\ P''_{ij},\ \cdots,\ P^{(k)}_{ij}, \tag{4}$$

in which each element after the first is obtained from its predecessor by the deletion of a closed cycle. The last element can involve no cycle, or is itself a cycle, and is called a *stem* of P_{ij}. The definition is not unique, since $P^{(k)}_{ij}$ clearly varies with the sequence of cycles of P_{ij} whose deletion leads to (4). Thus $a_{12}a_{23}a_{32}a_{24}a_{43}$ has the stems $a_{12}a_{23}$ and $a_{12}a_{24}a_{43}$.

From Lemmas 1 and 2 follows this lemma.

LEMMA 4. *A stem of a proper product P is a contraction of P which has no contraction. The order of a stem cannot exceed n.*

Increasing sequences. We first prove this lemma.

LEMMA 5. *For every integer c not exceeding p nor n, there occurs in M^p, $p \geqq 2$, an element involving a term which is a c-cyclic product.*

If $p = mc$ each diagonal element $a^{(p)}_{ii}$ involves certain c-cyclic terms in which a cycle C is repeated m times. If $p = mc + r$, $1 \leqq r < c$, each element $a^{(p)}_{ij}$, $i \neq j$, involves certain c-cyclic terms in which the rth factor, and hence the pth, is a_{kj}. Thus, for $n \geqq 4$, $a^{(4)}_{33}$ involves the 2-cyclic term $a_{31}a_{13}a_{31}a_{13}$, while $a^{(5)}_{24}$ involves the 3-cyclic term $a_{21}a_{14}a_{42}a_{21}a_{14}$.

THEOREM 5. *Let M be a set matrix of order n whose elements are independent. The sequence $M^{p_1}, M^{p_2}, \cdots, M^{p_r}, M^{p_{r+1}}, \cdots, p_r < p_{r+1}$, is increasing if and only if $p_1 > n-1$, while $p_{r+1} - p_r$ is a multiple of $1, 2, \cdots, n$.*

By Lemma 1 a term P of $a_{ij}^{(p_r)}$ involves a closed cycle C. Since $p_{r+1}-p_r$ is a multiple of c we can insert appropriately into P sufficient repetitions of C to yield by Lemma 3 an expansion $\overline{P}=P$ of order p_{r+1}. Thus $\overline{P}$ is the required term of $a_{ij}^{(p_{r+1})}$.

Conversely, if $p_1 \leqq n-1$ there is a term P' of $a_{ij}^{(p_1)}$ involving no cycle, and which by Lemma 1 and (A) is not contained in $a_{ij}^{(p)}$ for $p=p_2 \geqq n$. And if $p_1 > n-1$ while $p_{r+1}-p_r$ is not a multiple of $c \leqq n$, there is by Lemma 5 an element $a_{ij}^{(p_r)}$ of M^{p_r} involving a term P_r which is a c-cyclic product. Now a product P_{r+1} which contains P_r can by (A) involve only factors of P_r, and is hence c-cyclic. But since $p_{r+1}-p_r$ is not a multiple of c, it follows that P_{r+1} is not coterminal with P_r, and so is not a term of $a_{ij}^{(p_{r+1})}$. From (A) we conclude that P_r is not contained in the set $a_{ij}^{(p_{r+1})}$ and the theorem is established.

Decreasing sequences. Before proceeding we prove two lemmas.

LEMMA 6. *Let P be a proper product, S a sequence of cycles of P determining a stem P', $h=h(S)$ the highest common factor of the orders of the cycles of S, and c the greatest of these orders. There is a contraction $\overline{P}$ of P involving every subscript of P, and such that $\bar{p} \leqq c(n-c+2)-1$, $p \equiv \bar{p}$ (mod h).*

Consider the following sequence

$$S': \quad P', C_1, C_2, \cdots, C_k,$$

in which: (i) C_1 is a cycle of S involving a subscript not found in P', and has the maximum order of all such cycles. (ii) Each C following C_1 is a cycle of S involving a subscript which has previously appeared in S', and one which has not done so. Further, all such cycles of S are in the sequence S'.

It is easily shown that every subscript involved in P occurs in some cycle of S'. For let k_s be the first subscript of P not found in S'. Since k_s cannot occur in the stem P', it must first appear in P in a factor of the form $a_{k_r k_s}$. But k_s occurs in some cycle C' of S, while k_r occurs in a cycle of S'. We infer from (ii) that C' is a cycle of S', and a contradiction is reached.

Now if the cycles $C_1, C_2, \cdots, C_k$ exist, at least one involves a subscript of P'; for the contrary assumption leads to a contradiction, as in the above argument, on consideration of the first appearance in P of a subscript of $C_1, C_2, \cdots, C_k$. It follows that the cycles of S' can be combined with P' into a contraction $\overline{P}$ of P which involves every subscript of P.

Consider now the order of $\overline{P}$. We have

$$\bar{p} = p' + \sum_{t=1}^{k} c_t, \qquad c_t \leqq c.$$

If $p' \leqq c-2$, P' and C_1 together involve at least c distinct subscripts, whence $k \leqq n-c+1$, and $\bar{p} \leqq c-2+c(n-c+1) < c(n-c+2)-1$. While if $p' > c-2$, P' and C_1 together involve at least $p'+1$ distinct subscripts, whence $k \leqq n-p'$, and $\bar{p} \leqq p'+c(n-p') = cn-p'(c-1) \leqq cn-(c-1)^2 = c(n-c+2)-1$.

The congruence $p \equiv \bar{p} \pmod{h}$ follows from the definition of P' and $\bar{P}$. The lemma is established.

LEMMA 7. *Let $c_1 > c_2 > \cdots > c_m$ be a set S_1 of positive integers whose highest common factor is $h = h(S_1)$. The equation $c_1x_1 + c_2x_2 + \cdots + c_mx_m = k$ has a non-negative integral solution $(x_1', x_2', \cdots, x_m')$ for every integer k which is a multiple of h exceeding $(c_1c_2/h) - c_1 - c_2$.*[4]

THEOREM 6. *Let M be a set matrix of order n whose elements are independent. The sequence $M^{p_1}, M^{p_2}, \cdots, M^{p_r}, M^{p_{r+1}}, \cdots, p_r < p_{r+1}$, is decreasing if and only if $p_1 > (n-1)^2$, while $p_{r+1} - p_r$ is a multiple of $1, 2, \cdots, n$.*

Sufficiency. For any term P of $a_{ij}^{(p_{r+1})}$; the theorem asserts the existence of a term P_1 of $a_{ij}^{(p_r)}$ such that $P \subset P_1$. Our procedure is to insert into a contraction of P appropriate cycles of P of the precise total order required to yield the desired product P_1.

Let P' be a stem of P, determined by a sequence S of cycles of P. Let $\bar{P}$ be the contraction of P presented in Lemma 6; and let S_1: $C_1, C_2, \cdots, C_m$, be cycles of S among whose orders, $c_1 > c_2 > \cdots > c_m$ occur all orders of cycles of S. If $m=1$, the required term P_1 is clearly encountered in the sequence (4) of products defining P'.

Case 1. $c_1 < n$.

By Lemma 6 we have

$$\begin{aligned} p_r - \bar{p} &> (n-1)^2 - c_1(n - c_1 + 2) + 1 \\ &= c_1(c_1 - 1) - c_1 - (c_1 - 1) + (n-1)(n - c_1 - 1), \end{aligned}$$

whence

$$p_r - \bar{p} > c_1(c_1 - 1) - c_1 - (c_1 - 1) \geqq \frac{c_1c_2}{h} - c_1 - c_2. \tag{5}$$

Now by the same lemma, $p_{r+1} - \bar{p} \equiv 0 \pmod{h}$, hence from $p_{r+1} - p_r \equiv 0 \pmod{h}$ follows

[4] This lemma is readily established by mathematical induction. However, a better bound on k, namely, $(c_1 c_m/h) - c_1 - c_m$ has been communicated to the author by Dr. Alfred Brauer.

(6) $$p_r - \bar{p} \equiv 0 \pmod{h}.$$

By (5), (6), and Lemma 7 there is established the existence of non-negative integers, $x_1', x_2', \cdots, x_m'$ such that

$$\bar{p} + \sum_{t=1}^{m} c_t x_t' = p_r.$$

By Lemmas 6 and 3 we can insert the cycles $C_1, C_2, \cdots, C_m$ into $\bar{P}$ and obtain the required product P_1.

Case 2. $c_1 = n, c_2 < n-1$.

Again by Lemma 6

$$p_r - \bar{p} > (n-1)^2 - c_1(n - c_1 + 2) + 1$$
$$= c_1(c_1 - 2) - c_1 - (c_1 - 2) + (n-2)(n-c_1),$$

whence

$$p_r - \bar{p} > c_1(c_1 - 2) - c_1 - (c_1 - 2) \geqq \frac{c_1c_2}{h} - c_1 - c_2,$$

and the proof proceeds as in Case 1.

Case 3. $c_1 = n, c_2 = n-1$.

Here the contraction P', instead of $\bar{P}$, is employed. We have by Lemma 4

$$p_r - p' > (n-1)^2 - n = c_1c_2 - c_1 - c_2.$$

Since $h = 1$ it follows by Lemma 7 that non-negative integers, x_1', x_2', exist such that $p' + c_1x_1' + c_2x_2' = p_r$. Thus since P' must involve a subscript of C_1 and C_2, it is possible to insert the cycles C_1, C_2, into P' and obtain by Lemma 3 the required product P_1.

Necessity. As in Theorem 5 it can be shown that $p_{r+1} - p_r$ must be a multiple of $1, 2, \cdots, n$; while for the condition $p_1 > (n-1)^2$, it will suffice to produce a term of an element of M^{p_2} which is not contained in the corresponding element of M^{p_1}, although $p_2 - p_1$ is a multiple of $1, 2, \cdots, n$.

Consider the n-cyclic product P_1' of order $p_1 + n - 1$ whose defining cycle is $C_1 = a_{12}a_{23} \cdots a_{n1}$, and the $(n-1)$-cyclic product P_2' of order $p_2 - p_1 - n + 1$ whose defining cycle is $C_2 = a_{23}a_{34} \cdots a_{n2}$. By inserting P_2' into P_1' following any factor a_{12}, a proper product P of order p_2 is obtained which is a term of an element of the first row of M^{p_2}. Now from (A) and the structure of P it follows that any proper coterminal product containing P can be had from P by deletion of the cycles C_1, C_2. We are thus led to the equation

(7) $$nx_1 + (n-1)x_2 = p_2 - p_1,$$

with the restrictions

$$nx_1 \leqq p_1 + n - 1, \tag{8}$$

$$(n - 1)x_2 \leqq p_2 - p_1 - n + 1. \tag{9}$$

By (7) and (9), x_1 is a positive multiple of $n-1$, while from $p_1 \leqq (n-1)^2$ we have by (8), $nx_1 \leqq n(n-1)$. Thus $x_1 = n-1$, but it is clear that the deletion of $n-1$ cycles C_1 from P will yield a product[5] whose first factor is a_{23}. From (A) it follows that P is not contained in the corresponding element of M^{p_1}, and the theorem is established.

Equality of matrices. Theorems 5 and 6 provide conditions for increase and decrease, respectively, in a sequence of ascending powers of M. In summation we have the following theorem:

THEOREM 7. *Let M be a set matrix of order n whose elements are independent. The equality $M^{p_1} = M^{p_2}$, $p_2 > p_1$, holds if and only if $p_1 > (n-1)^2$, while $p_2 - p_1$ is a multiple of* $1, 2, \cdots, n$.

Theorem 4 is an immediate consequence.

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[5] Thus for $n=3$, $p_1=4$, $p_2=10$, we have $P = a_{12}a_{23}a_{32}a_{23}a_{32}a_{23}a_{31}a_{12}a_{23}a_{31}$ which has no contraction of order 4.