CLASSES OF MAXIMUM NUMBERS ASSOCIATED WITH TWO SYMMETRIC EQUATIONS

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1. **Introduction.** Let $\sum_{i,j}(1/x)$ stand for the elementary symmetric function of the jth order of the i reciprocals $(1/x_p)$ $(p=1, 2, \cdots, i>0)$ with

$$\sum_{i,j} (1/x) \equiv 0 \quad \text{when} \quad i < j \text{ or } j < 0,$$
$$\equiv 1 \quad \text{when} \quad j = 0$$

($\sum_{i,j}(x)$ having a similar meaning for the x_p themselves).

Here we extend the work of papers I,[1] II,[2] III[3] by obtaining relative to equations (1) and (1.1) below results analogous to those in I, II, III

$$(1)^4 \qquad \sum_{n,n-1} (1/x) + \sum_{i=1}^{m} a_i[\pi(x)]^{-i} = b/a,$$
$$a = (c+1)b - 1, \ \pi(x) = x_1 x_2 \cdots x_n,$$

$$(1.1) \qquad \sum_{n,n-2} (1/x) + \lambda \sum_{n,n-1} (1/x) + \mu \sum_{n,n} (1/x) = b/a;$$

in (1), b, c, and m are arbitrary positive integers, $n>1$, and the a_i are any non-negative real numbers; in (1.1), a and b are as in (1), $n>2$, λ is a non-negative integer, and μ is a positive integer.

We have not seen previous mention of (1); the case of (1.1) in which $\mu=0$ was treated in II and that in which $\lambda=\mu=1$ was treated in III. Our procedure for (1) does not suffice for the equation that is obtained by adding to the left member of (1.1) the terms

$$\sum_{i=2}^{m} a_i[\pi(x)]^{-i}.$$

The following definitions and notation from I will be frequently

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[1] H. A. Simmons, Transactions of this Society, vol. 34 (1932), pp. 876–907.

[2] Norma Stelford and H. A. Simmons, this Bulletin, vol. 40 (1934), pp. 884–894.

[3] H. A. Simmons and W. E. Block, Duke Mathematical Journal, vol. 2 (1936), pp. 317–340.

[4] In so far as we know, the form of the right member of equation (1) was first used by Tanzo Takenouchi in the Proceedings of the Physico-mathematical Society of Japan, (3), vol. 3 (1921), pp. 78–92.

used here: $x_{1\cdots p}$, $1 \leqq p \leqq n$, stands for the set $(x_1, x_2, \cdots, x_p)$; $P(x) \equiv P(x_1, x_2, \cdots, x_n)$ stands for a polynomial, not a constant, which is symmetric in the x_i $(i=1, 2, \cdots, n)$ and has at least one positive, and no negative, coefficient; the *Kellogg solution* of equation (e), where e stands for (1) or (1.1), is the solution that is obtained by minimizing the variables $x_1, x_2, \cdots, x_{n-1}$ in this order, one at a time, in (e) among positive integers; an *E-solution* of (e) is any solution of it in which $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ while $x_1, x_2, \cdots, x_n$ are positive integers. When any further definition or notation from I, II, or III is used here, a suitable reference to the appropriate article will be given.

We can now state accurately our purpose here. It is to prove that the Kellogg solution w of equation (e) has the following two properties, which were called *remarkable properties* in III: (i) It contains the largest number that exists in any E-solution of (e) and no other E-solution of (e) contains this number. (ii) If X, with $X \neq w$, is an E-solution of (e), then $P(X) < P(w)$.

The discussion from §2 to the end of this paper is divided into two parts as follows: Part 1 treats (1), §§2 to 6 (inclusive); Part 2, (1.1), §§7 to 12.

This paper involves innovations of notation and procedure of I, II, III. The terms *set* σ and *set* τ, which were important in I, II, III, are not used here; they are not needed because of our use of a new term that is very convenient for present purposes, namely *s-set* (cf. §4). This change is accompanied by new procedure for both (1) and (1.1): in Part 1, we use a new lemma, namely Lemma 4.1; in Part 2, we introduce an upper bound $R(X)$ (cf. §8) for the maximum number that we seek to identify and we show that $R(X)$ is uniquely maximized, with respect to values that $R(x)$ can assume on E-solutions X of (1.1), by the Kellogg solution of (1.1). In so far as we know, our reasoning about $R(X)$ in §11 affords the first strong resemblance of our procedure (for identifying maximum numbers) to that which Curtiss[5] used in solving Kellogg's problem.[6]

Part 1. The remarkable properties of the Kellogg solution of (1)

2. **The Kellogg solution of** (1). This solution is $x=w$ where [I, (23)]

$$(2) \qquad w_p = 1 \quad (p = 1, 2, \cdots, n-2), \qquad w_{n-1} = c + 1,$$

[5] D. R. Curtiss, American Mathematical Monthly, vol. 29 (1922), pp. 380–387, and note, in particular, his upper bound (10), p. 384.

[6] O. D. Kellogg, ibid., vol. 28 (1921), pp. 300–303.

with w_n defined to be the unique positive solution x_n of the equation that is obtained by substituting in (1) for each x_p ($p=1, 2, \cdots, n-1$) its value w_p from (2).

If $n=2$, only the last equation in (2) is to be retained.

3. **Our transformation.** In considering an E-solution $X \neq w$ of (1), we classify and transform elements as we did in §§15, 17 of I. Thus we define our transformation of X ($X_{1 \ldots n}$) into a new set X' by (t_1) or (t_2) [I, (33) and (52)]:

$$
(3) \qquad \begin{aligned}
&(t_1)\colon\ X'_p = X_p(p \neq q_{1,1}q;\ p \leqq n),\ X'_{q1} = w_{q1},\ Q(1/X') = Q(1/X);\\
&(t_2)\colon\ X'_p = X_p(p \neq q_{1,1}q;\ p \leqq n),\ X_{1q} = w_{1q},\ Q(1/X') = Q(1/X);
\end{aligned}
$$

according as (t_1) requires X'_{1q} to be not greater than w_{1q} or greater than w_{1q}, respectively, where $Q(1/X)$ is the case $x=X$ of the left member of (1); if (t_1) defines X'_{1q} to be equal to w_{1q}, (t_1) and (t_2) are the same transformation.

If $X' \neq w$, our transformation from X' to X'' is obtained from (3) by replacing in (3) X, X', q by X', X'', q', respectively, where $X'_{q'1}$ ($X'_{1q'}$) is of class $A'(B')$, and the new transformation is regarded as a transformation (3). Thus we avoid giving here an analogue of (52) of I.

Replacement of (1) by (1.1) in this section gives our transformation for §§11, 12; it will be called (3a).

4. **Important lemmas; s-set.** In the sequel, we use Lemma 4 and Lemma 4.1 below. Lemma 4.1 depends on Lemma 4, which is essentially Lemma 1a of I.

LEMMA 4. *Let $Q(1/x)$ stand for a symmetric polynomial in the n reciprocals $(1/x_p)$ ($p=1, 2, \cdots, n>1$) which is not a mere constant and contains at least one positive, and no negative, coefficient; with i and j equal to distinct positive integers each less than or equal to n, let x_i, x_j, α, β be positive numbers with $\alpha < x_i \leqq x_j$; and suppose that the expression that is obtained by replacing in $Q(1/x)$ the numbers x_i, x_j* by $(x_i-\alpha)$, $(x_j+\beta)$, *respectively, equals $Q(1/x)$, then*

$$
(4) \qquad x_i x_j \leqq (x_i - \alpha)(x_j + \beta), \quad x_i^h + x_j^h < (x_i - \alpha)^h + (x_j + \beta)^h,
$$

where h is a positive integer. Furthermore, the equality sign holds in (4) *if, and only if, $Q(1/x)$ is a polynomial in $[\pi(x)]^{-1}$.*

In the proof of Lemma 4.1 and in §§6, 12, we use the following definition.

DEFINITION OF s-set. *If in a set $X^{(\alpha)} \neq w$ every element of class $B^{(\alpha)}$* [I, p. 898] *is at least as large as every element of class $A^{(\alpha)}$, we call $X^{(\alpha)}$ an s-set (relative to w), s meaning satisfactory in the sense that $X^{(\alpha)}$ can be transformed into w by one or more transformations of type* (3) *every one of which accords with* (4).

LEMMA 4.1. *Let k be an integer greater than or equal to* 1; *let $W \equiv W_{1} \dots {}_{v}$ $(v>1)$ be the Kellogg solution of the equation $(x_1 x_2 \cdots x_v)^{-1} = k^{-1}$; and let $X \equiv X_{1} \dots {}_{v}$ be a set of v positive integers with $X_1 \leqq X_2 \leqq \cdots \leqq X_v$ satisfying the relation*

$$(4.1) \qquad (x_1 x_2 \cdots x_v)^{-1} \leqq k^{-1},$$

then

$$(4.2) \qquad \sum_{v,s} (1/X) \leqq \sum_{v,s} (1/W), \qquad 1 \leqq s < v,$$ [7]

with $<$ holding in (4.2) *except when $X = W$.*

PROOF. We first consider the case where

$$(4.3) \qquad X_1 X_2 \cdots X_v = k.$$

Then $X_{1} \dots {}_{v}$ is an s-set (relative to W). Therefore $P(X) \leqq P(W)$, [I, Lemma 3], the equality sign holding in this relation if, and only if, $X = W$ or $P(X)$ is a polynomial in the product of all of the v variables $X_1, X_2, \cdots, X_v$. In particular, then, when $P(x) \equiv \sum_{v,r}(x)$, we have

$$(4.4) \qquad \sum_{v,r} (X) \leqq \sum_{v,r} (W), \qquad 1 \leqq r < v,$$

the equality sign holding in (4.4) if, and only if, $X = W$ since $r < v$. But, using (4.3), we find

$$(4.5) \quad \sum_{v,r} (1/X) = \frac{\sum_{v,v-r} (X)}{X_1 X_2 \cdots X_v} = \frac{\sum_{v,v-r} (X)}{k}, \quad \sum_{v,r} (1/W) = \frac{\sum_{v,v-r} (W)}{k},$$

while, by (4.4),

$$(4.6) \qquad \sum_{v,v-r} (X) \leqq \sum_{v,v-r} (W), \qquad 1 \leqq v - r < v.$$

From (4.5) and (4.6), it follows that in the present case (4.2) holds.

Suppose now that $<$ holds in the case $x = X$ of (4.1), say

$$(4.7) \qquad (X_1 X_2 \cdots X_v)^{-1} = (k')^{-1},$$

[7] The case $s = v$ is excluded merely for convenience in our applications of Lemma 4.1; the case $k = 1$ is included for convenience in writing, not for use.

where k' is an integer greater than k. By considering the Kellogg solution of (4.7), one can prove that (4.2) still holds: indeed if the Kellogg solution of (4.7) is U, one finds readily that

$$\sum_{v,s}(1/X) \leqq \sum_{v,s}(1/U) < \sum_{v,s}(1/W), \qquad 1 \leqq s - v.$$

5. **Proof of Property** (i) **for the** w **of** (1). We substitute any E-solution $X \neq w$ of (1) for x in (1) and employ the following equivalent of the resulting equation ($\sum_{i,j}$ standing for $\sum_{i,j}(1/X)$ here as in the sequel)

$$(5) \qquad \sum_{n-1,n-1} + X_n^{-1}\left(\sum_{n-1,n-2} + a_1 \sum_{n-1,n-1}\right) + \sum_{i=2}^{m} a_i X_n^{-i}\left(\sum_{n-1,n-1}\right)^i = b/a.$$

In order to establish Property (i), it suffices to prove that w has Property (i) when $n=2$ and to prove that the following relations hold

$$(5.1) \qquad \sum_{n-1,n-1} \leqq \sum_{n-1,n-1}(1/w), \qquad \sum_{n-1,n-2} < \sum_{n-1,n-2}(1/w), \qquad n > 2.$$

When $n=2$, equation (1) reduces to

$$(5.2) \qquad X_1^{-1} + X_2^{-1}(1 + a_1 X_1^{-1}) + \sum_{i=2}^{m}(a_i X_1^{-i})X_2^{-i} = ba^{-1}.$$

In this case, $X \neq w$ implies that $X_1 < w_1$. This fact and (5.2) imply that $X_2 < w_2$ [I, Lemma 2].

When $n > 2$, the first relation of (5.1) is true by the definition of Kellogg solution since (cf. (2))

$$\sum_{n-1,n-1} \leqq (c+1)^{-1} = \sum_{n-1,n-1}(1/w),$$

and since $X \neq w$, the second relation of (5.1) follows from Lemma 4.1 (cf. the case of (4.2) in which $X \neq w$ and $(v, s) = (n-1, n-2)$ with $n > 2$).

6. **Proof that the** w **of** (1) **has Property** (ii). Let $X \neq w$ be an E-solution of (1). The discussion of the case $n=2$ in §5 shows that in this case X is an s-set (so that $P(X) < P(w)$). Suppose $n > 2$. Then, by §5, $X_n < w_n$, and by (2) every classified element of $X_1 \ldots _{(n-2)}$ is of class A. Therefore, whether X_{n-1} is of class A, class B, or unclassified ($=w_{n-1}$), X is an s-set.

Part 2. The remarkable properties of the Kellogg solution of (1.1)

7. **The Kellogg solution of** (1.1). This solution is $x = w$ where [I, (23)]

$$
(7)\qquad \begin{aligned}
&w_p = 1, && p = 1, 2, \cdots, n-3,\\
&w_{n-2} = c+1, \qquad w_{n-1} = a\Big[\sum_{n-2,1}(w) + \lambda\Big],\\
&w_n = a\Big[\sum_{n-1,2}(w) + \lambda\sum_{n-1,1}(w) + \mu\Big].
\end{aligned}
$$

If $n=3$, the first set of equations in (7) is, of course, to be omitted.

8. **An upper bound for X_n.** In the sequel X stands for an E-solution of (1.1), arbitrary except as we specify.

If we substitute X for x in (1.1) and solve the resulting equation for X_n, we find after simple algebraic manipulations that

$$
(8)\qquad \begin{aligned}
X_n = a&\Big[\sum_{n-1,2}(X) + \lambda\sum_{n-1,1}(X) + \mu\Big]\\
&\cdot\Big[bX_1X_2\cdots X_{n-1} - a\Big(\sum_{n-1,1}(X) + \lambda\Big)\Big]^{-1}.
\end{aligned}
$$

Since the X_p $(p=1, 2, \cdots, n-1)$ are positive integers, the second factor in the right member of (8) is the reciprocal of a positive integer. Therefore,

$$
(8.1)\qquad X_n \leqq a\Big[\sum_{n-1,2}(X) + \lambda\sum_{n-1,1}(X) + \mu\Big].
$$

9. **An inequality for X_{n-1} when X_n is the maximum number.** In the sequel, the statement that *$X \neq w$ and X_n is the maximum number that we seek to identify* (so that $X_n \geqq w_n$) will be referred to as *hypothesis H*, or merely as H. We use H henceforth until a contradiction of it is reached in §11.

Under hypothesis H, we now desire to prove that

$$
(9)\qquad X_{n-1} \leqq w_{n-1}.
$$

Suppose that this is not true, so that (with H holding)

$$
(9.1)\qquad X_{n-1} > w_{n-1}.
$$

We presently contradict (9.1). The case $x=X$ of (1.1) is equivalent to

$$
(9.2)\qquad \begin{aligned}
&\sum_{n-2,n-2} + (1/X_{n-1})\Big(\sum_{n-2,n-3} + \lambda\sum_{n-2,n-2}\Big)\\
&\qquad + (1/X_n)\Big(\sum_{n-1,n-3} + \lambda\sum_{n-1,n-2} + \mu\sum_{n-1,n-1}\Big) = b/a,\\
&\qquad\qquad a = (c+1)b - 1,
\end{aligned}
$$

with $n>2$. To reach the contradiction, we first establish the following relations

$$\sum_{n-2,n-2} \leqq \sum_{n-2,n-2} (1/w), \qquad n>2, \tag{9.3}$$

$$\sum_{n-2,n-3} +\lambda \sum_{n-2,n-2} \leqq \sum_{n-2,n-3} (1/w) + \lambda \sum_{n-2,n-2} (1/w), \qquad n>2. \tag{9.4}$$

Since X is an E-solution not equal to w of (1.1) and $n>2$, $X_1X_2 \cdots X_{n-2} \geqq c+1 = w_1w_2 \cdots w_{n-2}$; therefore, (9.3) is true. Consequently, to prove (9.4), it suffices to show that

$$\sum_{n-2,n-3} \leqq \sum_{n-2,n-3} (1/w), \qquad n>2. \tag{9.5}$$

When $n=3$, (9.5) states that $1=1$; when $n>3$, (9.5) is a case of Lemma 4.1 in which $X \neq W$ and $(v, s)=(n-2, n-3)$ with $n-3 \geqq 1$; therefore, (9.5) is true.

Next, using (9.1), (9.3), and (9.4), we find that the sum of the terms in the first line of (9.2) is less than U, where

$$U \equiv \sum_{n-2,n-2} (1/w) + (1/w_{n-1})\left[\sum_{n-2,n-3} (1/w) + \lambda \sum_{n-2,n-2} (1/w) \right].$$

Indeed, if in the first line of (9.2) we should replace X_{n-1} by $X_{n-1}-1$, the resulting expression would not exceed U. Consequently, there exists for (1.1) an E-solution Y in which

$$Y_p = X_p \ (p=1, 2, \cdots, n-2), \quad Y_{n-1} = X_{n-1} - 1, \quad Y_n > X_n, \tag{9.6}$$

and the inequality in (9.6) contradicts H. Hence, under hypothesis H, (9.1) is false.

10. **On the classification of the elements of $X_{1\cdots(n-1)}$ when H holds.** For use in §11, the following statement, S, will presently be proved: *In $X_{1\cdots(n-1)}$ every element of class B is at least as large as every element of class A.*

To avoid vacuous language in the proof of S,[8] we consider separately the cases $n=3$ and $n>3$.

Case $n=3$. Here $X_{1\cdots(n-1)} = (X_1, X_2)$, and either $X_p \geqq w_p$ $(p=1, 2)$ or $X_1 > w_1$, $X_2 < w_2$; in both cases S is true.

Case $n>3$. Here, by (7), any classified element of $X_{1\cdots(n-3)}$ is of class A; X_{n-2} is of class A, class B, or unclassified $(=w_{n-1})$; and by (9) X_{n-1} is either unclassified or of class B. Therefore S is true.

[8] If $n=3$, it is vacuous to say that any classified element of $X_{1\cdots(n-3)}$ is of class A (cf. our discussion of the case $n>3$).

11. **Proof of Property** (i) **for the** w **of** (1.1). If $X_{1\cdots(n-1)}$ contains no element of class B, $X\neq w$ implies that $X_n<w_n$, which contradicts H.

Suppose that $X_{1\cdots(n-1)}$ contains at least one element of class B and, therefore, at least one element of class A, since the first classified element of X is necessarily of class A. Then, by S, every application of transformation (3a) to X or to an *intermediate set* of X [I, p. 898], which does not change the magnitude of the nth element of a set, accords with (4). Further, the last such transformation in the *exhaustive set* for X [I, p. 898] yields a set $X^{(t)}$ in which $X_n^{(t)}=X_n\geqq w_n$ (cf. H) and

$$X_p^{(t)} \leqq w_p, \qquad p = 1, 2, \cdots, n-1; \tag{11}$$

otherwise $X^{(t)}$ would be a set satisfying (1.1) and having at least one element of class $A^{(t)}$ and no element of class $B^{(t)}$, which is impossible. We reach a contradiction of H as follows. Let $R(X)$ stand for the right member of (8.1), so that $X_n\leqq R(X)$. Certainly $R(X)$ is expressible in the form

$$R(X) = F + G(X_{q_1} + X_{1q}) + HX_{q_1}X_{1q},$$

in which F, G, H are positive and independent of X_{q_1} and X_{1q}, while $X_{q_1}\leqq X_{1q}$ [I, p. 898]; therefore, the first transformation (3a) that one uses in passing from X to $X^{(t)}$ is such that $R(X)<R(X')$ (cf. (4)). If $t>1$, our transformation of X' into X'' is such that $R(X')<R(X'')$, and so on. On arriving at $X^{(t)}$, one has

$$X_n \leqq R(X) < R(X') \leqq R(X^{(t)}), \qquad t \geqq 1. \tag{11.1}$$

But by (11) and the fact that $R(X^{(\alpha)})$ depends only on the first $n-1$ elements of $X^{(\alpha)}$, we have

$$R(X^{(t)}) \leqq R(w). \tag{11.2}$$

By (11.1) and (11.2), $X_n<R(w)$; and since $R(w)=w_n$ (cf. (7)), $X_n<w_n$. This contradicts H.

12. **Proof that the** w **of** (1.1) **has Property** (ii). H negated, we now merely suppose that $X\neq w$. We again avoid vacuous language by treating separately the cases $n=3$ and $n>3$.

If $n=3$, $X_3<w_3$ (cf. §11) and either $X_p\geqq w_p$ ($p=1$, 2) or $X_1>w_1$, $X_2<w_2$; in either case X is an s-set (and $P(X)<P(w)$).

If $n>3$, X_n is of class B, and every classified element of $X_{1\cdots(n-3)}$ is of class A (cf. (7)). Then if one of X_{n-2}, X_{n-1} is unclassified, X is an s-set; the same is true if X_{n-2}, X_{n-1} are of the same class or if

X_{n-2} (X_{n-1}) is of class A (B). Therefore, we only need to consider the case in which X_{n-2} (X_{n-1}) is of class B (A). Then

$$(12) \qquad X_1X_2 \cdots X_{n-2} \geqq w_1w_2 \cdots w_{n-2} = c + 1, \qquad n > 3,$$

and $X_{1\cdots(n-2)}$ contains one or more elements of class A (preceding the element X_{n-2}, of class B). Apply transformation (3a) to X, or to X and one or more intermediate sets of X, until a set $X^{(t)}$ is obtained in which $X_{1\cdots(n-2)}^{(t)}$ does not contain both an element of class $A^{(t)}$ and an element of class $B^{(t)}$. Since each transformation that has been applied to this point has increased the $(n-2)$d element of a set and decreased a not larger element (with subscript less than $n-2$) each transformation applied has accorded with (4), so that necessarily

$$X_1^{(t)}X_2^{(t)} \cdots X_{n-2}^{(t)} > c + 1$$

(cf. the first two lines below (4)), and

$$X_p^{(t)} \geqq w_p, \qquad p = 1, 2, \cdots, n-2,$$

with $>$ holding for at least one of the indicated values of p (cf. (12)). Further, by hypothesis $X_{n-1} > w_{n-1}$, and by §11 $X_n < w_n$, while no transformation used in arriving at $X^{(t)}$ has changed the value of the $(n-1)$th or nth element of a set. Therefore,

$$(12.1) \qquad X_p^{(t)} \geqq w_p \quad (p = 1, 2, \cdots, n-1), \qquad X_n^{(t)} = X_n < w_n,$$

with $>$ holding in (12.1) for at least one of the indicated values of p. If u is a value of p for which $>$ holds in (12.1), then $X_u^{(t)} \leqq X_u$ since transformation (3a) never increases the value of an element of class $A^{(\alpha)}$. Consequently, the classified elements of $X^{(t)}$, like the elements of X, do not decrease as their subscripts increase, and so the $X^{(t)}$ of (12.1) is an s-set.

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