rational canonical form is then easily obtained. Examples are given to support the contention that this process simplifies the computation of the rational canonical form of A. (Received December 15, 1941.)

ANALYSIS

108. G. E. Forsythe and A. C. Schaeffer: *A remark on Toeplitz matrices.*

A doubly infinite matrix (a_{mn}) is said to be regular if for every sequence $\{x_n\}$ with limit x' the corresponding sums $y_m=\sum_n a_{mn}x_n$ are defined for all m and have the limit x'. An apparently more general definition of regularity is that the sums defining y_m exist for all sufficiently large m, depending on $\{x_n\}$, and have the limit x'. Tamarkin (this Bulletin, vol. 41 (1935), pp. 241–243) has obtained necessary and sufficient conditions for the second type of regularity. This result is obtained by elementary methods and related topics are discussed. (Received January 23, 1942.)

109. H. L. Garabedian: *Hausdorff integral transformations.*

This paper involves a study of the integral transformation $v(x)=\int_0^x u(y)d\phi(y/x)$, defining a method of summation $(H, \phi(x))$, where $u(x)$ is bounded and continuous, $x\geqq 0$, and where $\phi(x)$ is either a Hausdorff mass function or satisfies the conditions: (i) $\phi(x)$ is of bounded variation on the interval $0\leqq x\leqq 1$, (ii) $\phi(x)$ is continuous on the interval $(0, 1)$ except possibly at $x=1$, (iii) $\phi(0)=0$, (iv) $\phi(1)=1$. It is proved that the transformation is regular when and only when $\phi(x)$ is a Hausdorff mass function, and sufficient conditions involving the Silverman-Schmidt integral equations are obtained in order that $(H, \phi_1(x))\supset(H, \phi_2(x))$, in the case that $\phi_1(x)$ and $\phi_2(x)$ satisfy the conditions stated above. These results are extensions of those obtained by Silverman (Transactions of this Society, vol. 26 (1924), pp. 101–112). (Received January 10, 1942.)

110. A. M. Gelbart: *Functions of two variables with bounded real parts in domains not equivalent to the bicylinder.*

Let $f(z_1, z_2)$ be regular in the interior of a finite four-dimensional domain $\mathfrak{M}^4$, bounded by certain analytic hypersurfaces, and in general not equivalent to the bicylinder, and let $f(z_1, z_2)$ have a bounded real part in $\mathfrak{M}^4$. These domains were first considered by Bergman, and are termed by him, domains with distinguished boundary surfaces. An upper bound for $|f(z_1, z_2)|$ is obtained in terms of only max $Re\, f(z_1, z_2)$ in $\mathfrak{M}^4$, $f(0, 0)$ and the domain considered. From a formula for $\partial^{m+n}f(z_1, z_2)/\partial z_1^m\partial z_2^n$ in $\mathfrak{M}^4$, previously obtained by the author (Transactions of this Society, vol. 49 (1941), pp. 199–210), an upper bound is also obtained for $|\partial^{m+n}f(z_1, z_2)/\partial z_1^m\partial z_2^n|$, again in terms of only max $Re\, f(z_1, z_2)$ in $\mathfrak{M}^4$, $f(0, 0)$ and the domain. These results depend upon the establishment of a form of the Schwarz lemma in $\mathfrak{M}^4$ for two variables. (Received January 29, 1942.)

111. H. J. Greenberg and H. S. Wall: *Hausdorff means included between $(C, 0)$ and $(C, 1)$.*

It is shown that if $\phi(u)$ is any function of bounded variation on the interval $0\leqq u\leqq +\infty$ such that $\phi(+\infty)-\phi(0)=1$, then the function $\alpha(z)=\int_0^\infty d\phi(u)/(1+zu)$ is a regular moment function; and that when $\phi(u)$ is further restricted to be monotone then $(C, 0) \subset [H, \alpha(n)] \subset (C, 1)$. Conditions under which $[H, \alpha(n)]$ is equivalent to $(C, 0)$ or to $(C, 1)$ are obtained which are analogous to the conditions found by Scott and Wall (abstract 47-3-144) for the special case where $\phi(u) \equiv 1$ for $u \geqq 1$, $\phi(0)=0$, namely $(C, 0) \approx [H, \alpha(n)]$ if and only if $\phi(+0)-\phi(0)>0$, and $[H, \alpha(n)] \approx (C, 1)$ if and only if $\int_0^\infty d\phi(u)/u < \infty$. Certain transformations of moment sequences, for example, the Hausdorff transformation, are discussed (Received January 14, 1942).

112. Walter Leighton and W. J. Thron: *On the convergence of continued fractions.*

In the $z=x+iy$ plane let new coordinate axes x', y' be obtained by rotating the original axes through an angle β. It is shown that if the elements a_n of the continued fraction $1+K(a_n/1)$ are complex numbers and lie in a closed bounded region in the interior of one of the parabolic regions $y'^2 \leqq \cos^2\beta/2(x'+(1/4)\cos^2\beta/2), -\pi<\beta<\pi$, the continued fraction converges. Further it is established that the value of this continued fraction lies in the half-plane defined by the relation $(x-1/2)\cos\beta/2+y\sin\beta/2 \leqq 0$. (Received January 8, 1942.)

113. Walter Leighton and W. J. Thron: *On value regions of continued fractions.*

If the elements $a_n=\rho e^{ip}$ of a continued fraction $1+K(a_n/1)$ lie in a parabolic region $\rho \leqq 2d(1-d)/(1-\cos\theta)$ $(1/2<d<1)$ and if the a_n are bounded in absolute value, the continued fraction is known to converge. It is shown, that the value $z=Re^{i\theta}$ of this continued fraction lies in the region $R \geqq 2d(1-d)/(1-2d+\cos\theta)$, $-\beta<\theta<\beta$; $\beta=\arccos(1-2d)$. Every value in this region is taken on by at least one continued fraction $1+K(a_n/1)$ with elements in the described parabola. (Received December 12, 1941.)

114. A. N. Lowan, Gertrude Blanch, and William Horenstein: *Inversion of the q-series associated with Jacobi elliptic functions.*

In the computation of the Jacobi elliptic functions, sn, cn, and dn with the aid of the theta functions, it was found necessary to invert the expression $\epsilon=(1/2)(1-k'^{1/2})/(1+k'^{1/2})=(1/2)\theta_2(0, q^4)/\theta_3(0, q^4)$, and thus obtain q as a power series of ϵ. Weierstrass had given the first four terms of this expansion (*Werke*, II (1895), p. 276); Milne-Thomson found two additional terms (Journal of the London Mathematical Society, vol. 5 (1930), pp. 148–149). The authors have found the first fourteen terms in the desired expansion. This makes it possible to compute q in terms of ϵ with an accuracy varying between seven places for $\epsilon=0.4$ to eighteen places or better for $\epsilon \leqq 0.25$. These results were obtained in the course of work by the Mathematical Tables Project, Work Projects Administration for the City of New York, conducted under the sponsorship of the National Bureau of Standards. (Received December 17, 1941.)

115. A. N. Lowan and Abraham Hillman: *A short table of the zeros of the equation $f(x)=J_0(x)Y_0(kx)-J_0(kx)Y_0(x)=0$.*

The Mathematical Tables Project conducted by the Work Projects Administration of New York City, under the sponsorship of the National Bureau of Standards, has computed a short table of the first five zeros of the above equation for $k=1$ (0.5) 4.0. The zeros were first computed with the aid of the method of McMahon, Annals of Mathematics, vol. 9 (1894–1895), pp. 23–30. Each zero was recomputed by inverse interpolation from the values of $f(x)$ for about six arguments at intervals of 0.01 in the neighborhood of the zero in question. As was anticipated from the fact that McMahon's formulae are based on the asymptotic expansions of Bessel functions, the accuracy of the zeros computed by the latter method increases with the order of the zeros. To illustrate, for $k=3$, the first zero is correct to only two decimal places whereas the fifth zero is correct to six places. When each one of the zeros has been computed for a number of k's at sufficiently small intervals, it is expected that the zero in question for any value of k within the given range will be easily obtainable by interpolation. (Received December 6, 1941.)

116. K. L. Nielsen: *Some properties of functions satisfying partial differential equations of elliptic type.*

The author considers the totality, T, of particular solutions of a partial differential equation, $L(u) \equiv \Delta u + a_1\partial u/\partial x + a_2 \partial u/\partial y + a_3 u = 0$, $a_k = a_k(x, y)$, $[k=1, 2, 3]$. Using results of operators transforming analytic functions into solutions of $L(u)=0$; [see Bergman, Comptes Rendus de l'Academie des Sciences, Paris, vol. 205 (1937), p. 1360 and Matematicheskii Sbornik, vol. 44 (1937), p. 1169], it is shown that for certain types of L there is a subclass S of functions belonging to T with the property that there exists a denumerable set $u_n(x, y) \in S$ each of which satisfies an ordinary differential equation, $\sum_{\nu=1}^{k} A_\nu(x, y; n) d^k u(x, y)/dx^k$ (A_ν being algebraic functions of x and y and depending on n in a simple way), and an analogous equation with respect to y. Every $u \in S$, regular in $x^2+y^2 \leqq \rho^2$, can be expanded in the uniformly convergent series $\sum \alpha_n u_n(x, y)$ in this circle. The singularities of $u \in S$ may be branch points of the type that u can be decomposed into $u = u_1 + u_2$, where u_2 is regular at the singular point and u_1 satisfies an ordinary differential equation analogous to the one above. If the function element $u = \sum A_{mn} x^m y^n$, $u \in S$, is given, the author indicates a procedure to determine from A_{mn} whether u has only singularities described above, and for the determination of the distances of these branch points from the origin. (Received January 29, 1942.)

117. Harry Pollard: *The generalized Stieltjes transform.*

In this paper the author studies the extension to the generalized Stieltjes transform (1) $f(x) = \int_0^\infty (x+t)^{-\rho} d\alpha(t)$ of the theory developed by Widder for the case $\rho = 1$. Specifically, the following results are obtained: (i) It is established that $f(x)$ is also a Laplace-Stieltjes transform for positive x if and only if $\alpha(t) = o(t^\rho)$ as $t \to \infty$. (ii) The inversion of (1) is accomplished by means of a linear differential operator. (iii) Necessary and sufficient conditions are obtained for the representation of a function $f(x)$ in the most general form (1). (iv) Conditions are obtained for the representation of $f(x)$ in the form (1) with $\alpha(t)$ of preassigned type. The results (i) and (iii) are new even for $\rho = 1$. (Received January 30, 1942.)

118. Raphaël Salem: *On singular monotonic functions of the Cantor type.*

The first part of the paper gives the construction of a singular monotonic function *of the Cantor type* with Fourier-Stieltjes coefficients of order $n^{(-1/2)+\epsilon}$ ($\epsilon > 0$ as small as desired) and even $n^{-1/2}\Omega(n)$, $\Omega(n)$ increasing to infinity as slowly as desired. The second part gives some new examples of sets of uniqueness and sets of multiplicity for trigonometrical series and shows that a very simple mapping can transform a set of uniqueness into a set of multiplicity. The third part gives the construction of a continuous monotonic function $F(x)$ of the Cantor type such that $\int_0^{2\pi}\exp(n_p ix)dF$ tends to $F(2\pi)-F(0)$ for a sequence $\{n_p\}$ such that p/n_p tends to zero as slowly as desired. (Received January 27, 1942.)

119. A. C. Schaeffer: *On the oscillation of differential transforms*. III.

It is shown that if in an interval (a, b) all derivatives of a function exist and no derivative changes sign more than a fixed bounded number of times in the interval then the function is analytic in the interval. This answers a question which was raised by Pólya as a generalization of a theorem of S. Bernstein. In the case in which all derivatives of a function $f(x)$ exist for $-\infty < x < \infty$ and the function is of exponential growth, let $g(n)$ be the maximum number of variations in sign of $f^{(n)}(x)$ in any interval of length, say, 1. If $g(n)$ tends to infinity with n, but sufficiently slowly, it can be shown that $f(x)$ is an entire function of order 1. These results are obtained by first showing that there is a function $p(n)$ such that the following statement is true: if $f(x)$ is bounded by 1 in $(-1, 1)$ and its first n derivatives are continuous in this interval, then the inequality $|f'(0)| > p(n)$ implies that $f^{(n)}(x)$ changes sign at least $n-1$ times in the interval. The preceding papers in this series have been written by G. Szegö and Einar Hille. (Received January 28, 1942.)

120. L. L. Silverman and Otto Szász: *On a class of Nörlund matrices.*

The definition of a Nörlund matrix depends upon a sequence of numbers p_n. The Nörlund matrix is then a triangular matrix, whose elements are the numbers p_{n-k} divided by the sum of the numbers p_n from zero to n. A Nörlund matrix is defined to be of *finite rank* if $p_n \neq 0$ for some value of n, and $p_n = 0$ for all greater n. A matrix of finite rank is *simple* if the numbers p_n are all zero or unity. In this paper some general properties of matrices of finite rank, and of simple matrices are obtained. A simple matrix for which *all* the numbers $p_n = 1$, when $n < r$, and for which $p_n = 0$ when $n \geqq r$ is called a Z-matrix. It is denoted by Z_r. The relative inclusion of corresponding summability methods among themselves and with the arithmetic mean methods is investigated. Among the results obtained are the following: if h is a factor of k then Z_h is included in Z_k; if h is prime to k, then the only sequences evaluated by both definitions are the convergent sequences. A study is also made of the inverse if the transformation which is a linear combination of the identity and Z_k. (Received January 22, 1942.)

121. Wolfgang Wasow: *On boundary layer problems in the theory of ordinary differential equations.*

Given a differential equation involving a parameter ρ in such a way that when ρ tends to infinity a "limiting" differential equation of lower order than the original one is obtained. What happens then to the solution $U(x, \rho)$ of a boundary value problem of the differential equation, if ρ tends to infinity? For ordinary linear differential equations of order n depending linearly on the parameter, and for a rather general class of boundary conditions not involving ρ, this question is answered in the paper by an easily applicable rule. This rule shows that in general $u(x) = \lim_{\rho\to\infty} U(x, \rho)$ exists only if the n boundary conditions are not too unevenly divided between the two end points. If the limit function exists it is a solution of the limiting differential equation but satisfies no longer all the boundary conditions prescribed for $U(x, \rho)$. The rule tells which of the n boundary conditions cease to be satisfied after the passage to the limit.

The proof consists in an asymptotic calculation of $U(x, \rho)$ based on the theory of asymptotic solution of ordinary linear differential equations involving a parameter, as developed by G. D. Birkhoff, Noaillon, Tamarkin, Trjitzinsky and others. (Received January 8, 1942.)

Applied Mathematics

122. Stefan Bergman: *Two-dimensional flow around two profiles.*

The study of the influence of tail surfaces on lift and pressure distribution of a wing can be reduced to the investigation of problems in the conformal mapping of doubly connected domains. The use of orthogonal functions enables one to give simple formulas for the lift and the moment in the case of a uniform flow around one and around two profiles. In the first case the lift, L, is found to be the expression $L=4(\pi)^{1/2}\rho V^2 (\sum|\phi_\nu(0)|^2)^{1/2} \sin [\alpha+\pi-\arg(\sum^*\psi_\nu(b)\phi_\nu(0))]$, where ρ is the density, $Ve^{-i\alpha}$ the velocity at infinity, $\psi_\nu(z)=\int_0^z\phi_\nu(z)dz$, and $\{\phi_\nu(z)\}$ a complete system of orthonormal polynomials of a domain B. (B is the domain obtained from the exterior of the profile by the transformation $z=1/\zeta$, b is the coordinate of the cusp, and the summation $\sum^*$ is understood in a certain special sense.) Analogous formulas exist for the moment and similar ones in the case of a flow around two profiles. (See also Notes of Lectures on Conformal Mapping, publication of Brown University, chap. XI, §§5–7) (Received January 29, 1942.)

123. Henry Wallman: *On the reduction in harmonic distortion due to high frequency pre-emphasis.* Preliminary report.

It is now common practice in high-fidelity sound broadcasting, in either FM or AM, to employ high frequency pre-emphasis, the object being an increase in signal-to-noise ratio. An additional effect, namely a reduction in harmonic distortion, has been noted experimentally. An analysis of this reduction in distortion is made in this paper, and quantitative evaluations are given for single-tone harmonic distortion of all orders. (Received December 30, 1941.)

124. Alexander Weinstein: *Spherical pendulum and complex integration.*

The following theorem, due to Puiseux (Journal de Mathématiques, 1842) is proved by a simple application of the theory of residues: The increment of the azimuth of a spherical pendulum corresponding to its passage from the lowest level z_1 to the highest level z_2 is greater than $\pi/2$. The boundary of the domain in the complex z-plane to which Cauchy's theorem is applied consists of a cut connecting z_1 with z_2 and of a vertical straight line to the right of z_2. (Received January 26, 1942.)

Geometry

125. P. O. Bell: *The parametric osculating quadrics of a family of curves on a surface.*

In this paper the author investigates the properties of the parametric osculating quadrics of a family of curves on a surface. These quadrics were introduced by Dan Sun (Tôhoku Mathematical Journal, vol. 32 (1930), pp. 81–85). His definition is essentially the following: At three neighboring points P, P_1, P_2 on an asymptotic curve C_u of a surface S construct the tangents to the curves of a one-parameter family on S.