NOTE ON THE COEFFICIENTS OF OVERCONVERGENT POWER SERIES

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M. B. Porter gave the first known example of an overconvergent power series, that is to say, of a power series in the complex variable with finite radius of convergence such that a suitable sequence of partial sums converges uniformly in a region containing in its interior both points inside and points outside the circle of convergence. Bourion has recently published[1] a general exposition of the theory of overconvergence to which the reader is referred for further historical and technical details.

Ostrowski established the surprising result that a power series $\sum_{n=0}^{\infty} a_n z^n$ of which the partial sums $s_{m_k} \equiv \sum_{n=0}^{m_k} a_n z^n$ exhibit overconvergence, can be expressed as the sum of a power series $\sum_0^{\infty} a_n' z^n$ with a larger radius of convergence and a power series of the form

$$(1) \qquad \sum_0^{\infty} a_n'' z^n, \qquad a_n'' = 0, \quad \text{whenever } m_k < n \leqq n_k$$

where n_k and λ are suitably chosen, with $m_k < \lambda n_k$, $0 < \lambda < 1$. Here we have $a_n = a_n' + a_n''$, $a_n' \cdot a_n'' = 0$; the partial sums $s_{m_k}''(z) = \sum_{n=0}^{m_k} a_n'' z^n$ of (1) also exhibit overconvergence.

It is the object of the present note to employ methods already known in the literature to make Ostrowski's result slightly more precise, especially to indicate that in series (1) the gaps cannot be uniquely defined with abrupt initial and terminal elements impossible of alteration by Ostrowski's process of writing the series as the sum of a series with a larger radius of convergence and a series with larger gaps which exhibits overconvergence. The moduli of the coefficients a_n'' must taper off gradually before the gap (m_k, n_k), and must increase gradually after the end of the gap; this remark is to be understood first in the sense that there is an upper limit to the moduli of the coefficients near the ends of a gap, a limit which increases as one moves away from the gap.

THEOREM 1. *Let the series*

(2)
$$\sum_{n=0}^{\infty} c_n z^n$$

whose radius of convergence is unity:

(3)
$$\limsup_{n \to \infty} |c_n|^{1/n} = 1,$$

have the gaps (m_1, n_1), (m_2, n_2), $\cdots$ *in the sense that* $c_n = 0$ *whenever* $m_k < n \leq n_k$, *and let the sequence of partial sums* $s_{m_k}(z) \equiv \sum_{n=0}^{m_k} c_n z^n$ *exhibit overconvergence. If* $R_0 > 1$ *is arbitrary, there exists* σ *depending on* R_0 *with* $0 < \sigma < 1$ *such that*

(4)
$$\limsup_{\mu_k \to \infty} \left[|c_{\mu_k}|, \ \mu_k \leq m_k \right]^{1/\mu_k} \leq R_0^{\sigma(\limsup m_k/\mu_k) - 1};$$

if $r_0 < 1$ *is arbitrary, there exists* τ *depending on* r_0 *with* $\tau > 1$ *such that*

(5)
$$\limsup_{\nu_k \to \infty} \left[|c_{\nu_k}|, \ \nu_k > n_k \right]^{1/\nu_k} \leq r_0^{\tau(\liminf n_k/\nu_k) - 1}.$$

The only novelty in Theorem 1 is its emphasis on (4) and (5) for the series (2) which overconverges and which possesses gaps, rather than for a series which overconverges and into which gaps may be introduced by Ostrowski's process; compare Bourion loc. cit., chap. 1, §2.

With the general notation

$$f(z) \equiv \sum_0^{\infty} c_n z^n, \qquad s_m(z) \equiv \sum_0^m c_n z^n, \qquad r_m(z) \equiv f(z) - s_m(z),$$

Cauchy's inequality yields

(6) $\left[\max |s_n(z)|, \text{ for } |z| = R_0 > 1 \right] \geq |c_0|, \ |c_1| R_0, \cdots, |c_n| R_0^n;$

(7) $\left[\max |r_n(z)|, \text{ for } |z| = r_0 < 1 \right] \geq |c_{n+1}| r_0^{n+1}, \ |c_{n+2}| r_0^{n+2}, \cdots.$

Under the hypothesis of Theorem 1 we have for suitable σ and τ (these inequalities follow from the fact of overconvergence by the use of a suitable harmonic majorant)

(8)
$$\limsup_{m_k \to \infty} \left[\max |s_{m_k}(z)|, \text{ for } |z| = R_0 \right]^{1/m_k} = R_0^{\sigma},$$

(9)
$$\limsup_{n_k \to \infty} \left[\max |r_{n_k}(z)|, \text{ for } |z| = r_0 \right]^{1/n_k} = r_0^{\tau}.$$

By virtue of (6) and (8) we have

$$(10) \qquad \limsup_{m_k \to \infty} \left[\left| c_{\mu_k} \right| \cdot R_0^{\mu_k}, \ \mu_k \leqq m_k \right]^{1/m_k} \leqq R_0^{\sigma},$$

which implies (4); by virtue of (7) and (9) we have

$$(11) \qquad \limsup_{n_k \to \infty} \left[\left| c_{\nu_k} \right| \cdot r_0^{\nu_k}, \ \nu_k > n_k \right]^{1/n_k} \leqq r_0^{\tau},$$

which implies (5). Theorem 1 is established. The first member of (4) is less than unity so long as we have $\limsup m_k/\mu_k < 1/\sigma$, and the first member of (5) is less than unity so long as we have $\liminf n_k/\nu_k > 1/\tau$.

A further description of the tapering-off of the moduli of the coefficients can be elaborated as follows. Under the conditions of Theorem 1, there exists a sequence c_{p_k} with $\lim_{p_k \to \infty} \left| c_{p_k} \right|^{1/p_k} = 1$; if necessary we change the notation of m_k, n_k, p_k so that we have also $m_1 < n_1 < p_1 < m_2 < n_2 < p_2 < \cdots$. It is now more convenient to employ (10) rather than (4); by setting $\mu_k = p_{k-1}$ we find

$$\liminf m_k/p_{k-1} \geqq 1/\sigma;$$

consequently the numbers $m_k - p_{k-1}$ cannot be small relative to p_{k-1}. In a similar manner we find from (11) with $\nu_k = p_k$

$$\limsup n_k/p_k \leqq 1/\tau;$$

consequently the numbers $p_k - n_k$ cannot be small relative to p_k. It will be noticed that with our present notation the moduli of the coefficients do taper off from the $\left| c_{p_k} \right|$, at least immediately before and after the gaps, because the second members of (4) and (5) are less than unity for $\mu_k = m_k$ and for $\nu_k = n_k + 1$. But we have not shown, nor is it true, that the moduli of the coefficients necessarily taper off monotonically.

As an application of Theorem 1 we prove (compare Bourion, chap. 2, §4) the following theorem:

THEOREM 2. *Let the series* (2) *have the radius of convergence unity, so that* (3) *is satisfied. Let unity be an isolated limit point of the set* $\left\{ \left| c_n \right|^{1/n} \right\}$. *Let one of the following conditions be satisfied:*

(a) *the series has gaps of relative lengths bounded from zero, in the sense that* $c_n = 0$ *whenever* $m_k < n \leqq n_k$, *with* $m_k < \lambda n_k$, $\lambda < 1$;

(b) *for some* $R_0 > 1$ *and* σ, $0 < \sigma < 1$, *and for some sequence* m_k, *equation* (8) *is valid;*

(c) *for some* $r_0 < 1$ *and* $\tau > 1$ *and for some sequence* n_k, *equation* (9) *is valid.*

Then the unit circle is a natural boundary for the series (2).

If the unit circle is not a natural boundary for the series (2), the function $f(z)$ represented is analytic along some arc A of the unit circle, and Ostrowski has shown that the conditions (a), (b), (c) imply overconvergence of the respective sequences $s_{m_k}(z)$, $s_{m_k}(z)$, $s_{n_k}(z)$ across the arc A.

Let us suppose that no limit point of the sequence $\{|c_n|^{1/n}\}$ other than unity lies in some interval $(1, 1-\eta)$, $\eta > 0$; we set

$$c_n' = c_n, \qquad\qquad \text{if } |c_n|^{1/n} > 1 - \tfrac{1}{2}\eta,$$

$$c_n' = 0, \qquad\qquad \text{if } |c_n|^{1/n} \leq 1 - \tfrac{1}{2}\eta,$$

$$c_n'' = c_n - c_n',$$

$$f_1(z) \equiv \sum_0^\infty c_n' z^n, \qquad f_2(z) \equiv \sum_0^\infty c_n'' z^n, \qquad f(z) \equiv f_1(z) + f_2(z).$$

The series defining $f_2(z)$ has a radius of convergence greater than unity; any overconvergent sequence for $f(z)$ is an overconvergent sequence for $f_1(z)$; it follows from Theorem 1 that $f_1(z)$ has no overconvergent sequence. Theorem 2 is established.

A necessary condition that a series (2) with (3) satisfied exhibit overconvergence is therefore that unity be a non-isolated limit point of the set $\{|c_n|^{1/n}\}$.

It is instructive in considering Theorem 1 to compare such an example as $\sum [z(z+1)]^{3^n}$, suggested by Bourion as a special case of Porter's original formulas; the function represented has the lemniscate $|z(z+1)| = 1$ as a natural boundary; the Taylor development about the origin is convergent in the circle $|z| < \tfrac{1}{2}5^{1/2} - \tfrac{1}{2} = 0.6$; the coefficients exhibit the characteristics described in Theorem 1.

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