ON CONVEX SETS IN LINEAR NORMED SPACES

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M. Eidelheit has proved[1] this theorem.

THEOREM. *In a linear normed space two convex bodies (that is, convex sets with inner points) having no common inner points are separated*[2] *by a plane.*

The purpose of this note is to present a quite different and somewhat simpler proof of this result.[3]

It is known[4] for linear normed spaces that

(1) *Through every boundary point of a convex body there passes a plane supporting the body.*

A convex cone with the point x_0 as vertex is defined as a convex body C containing at least one point $x \neq x_0$ and such that for each such point x in C,

$$ax + (1 - a)x_0 \in C, \qquad a \geqq 0.$$

It is easily seen that

(2) *Every supporting plane of a convex cone C passes through the vertex x_0 of the cone.*

For, let $L(x) - b = 0$, where $L(x)$ is a linear functional and b is a constant, define a plane of support of C passing through a boundary point y of C. Suppose for definiteness that

$$L(x) - b \leqq 0$$

holds for all points x in C. Then since every point of the form $ay + (1-a)x_0$ $(a \geqq 0)$ is a boundary point of C,

$$L(ay + (1 - a)x_0) - b \leqq 0, \qquad a \geqq 0,$$

Presented to the Society, September 5, 1941; received by the editors May 6, 1941.

[1] M. Eidelheit, *Zur Theorie der konvexen Mengen in linearen normierten Räumen*, Studia Mathematica, vol. 6 (1936), pp. 104–111.

[2] Two sets are separated by a plane provided they lie in opposite closed half-spaces of the plane.

[3] Added in proof: There has recently been brought to my attention another proof of Eidelheit's theorem by S. Kakutani, Proceedings of the Imperial Academy of Japan, vol. 13 (1937), pp. 93–94. The first part of the present proof is closely related to the first part of Kakutani's proof.

[4] See S. Mazur, *Über konvexen Mengen in linearen normierten Räumen*, Studia Mathematica, vol. 4 (1933), p. 74.

whence by the linearity of L

$$a(L(y) - b) + (1 - a)(L(x_0) - b) \leqq 0, \qquad a \geqq 0.$$

But $L(y) - b = 0$. Hence

$$(1 - a)(L(x_0) - b) \leqq 0, \qquad a \geqq 0,$$

and since the factor $(1-a)$ can change sign, we must have

$$L(x_0) - b = 0.$$

Let K_1 and K_2 be convex bodies with no common inner points. Let x_1 and x_2 be inner points of K_1 and K_2, respectively. There are unique boundary points x_1' and x_2' of K_1 and K_2 on the segment $\overline{x_1x_2}$. Consider the (perhaps degenerate) segment

$$L \equiv \overrightarrow{x_1' x_2'}.$$

Let L_1 be the set of all points $x \in L$ such that no segment joining x to an inner point of K_1 contains an inner point of K_2. The set L_1 is non-empty, since $x_1' \in L_1$. Furthermore, since the complement of L_1 in L is clearly open in L, L_1 is closed in L. Likewise, if L_2 is analogously defined, L_2 is a non-empty set closed in L.

Suppose there exists a point $x \in L - (L_1 + L_2)$. Then it is possible to join x to inner points y_1 and y_2 of K_1 and K_2 such that there exist inner points z_1 and z_2 of K_1 and K_2 lying on the (open) segments

$$\overline{xy_2}, \qquad \overline{xy_1},$$

respectively. But then the segments

$$\overline{y_1z_1}, \qquad \overline{y_2z_2}$$

intersect in a point z which is interior to both K_1 and K_2, contradicting the hypothesis. Hence the supposition that $L - (L_1 + L_2)$ is non-empty is false, and $L = L_1 + L_2$.

It now follows from the connectedness of L that there exists a point $x_0 \in L_1 \cdot L_2$. Let C_1 and C_2 be the sets consisting of all points on rays from x_0 through the inner points of K_1 and K_2, respectively.

(3) *The sets C_1 and C_2 are convex cones having no common points except x_0.*

The fact that C_1 and C_2 are convex cones is evident. If C_1 and C_2 had a common point $y \neq x_0$, then on the ray from x_0 through y there would have to be a point interior to K_1 and a point interior to K_2, contradicting $x_0 \in L_1 \cdot L_2$.

Take the point x_0 to be the origin θ. Let C_1^- denote the reflection in the point θ of the cone C_1: that is, C_1^- is the set of all points of the form $-x$, $x \in C_1$. The set C_1^- is a convex cone with θ as vertex. Let C denote the set of all points of the form

$$ax + (1 - a)y,$$

where $x \in C_2$, $y \in C_1^-$, and $0 \leqq a \leqq 1$.

(4) *The set C is a convex cone with θ as vertex. Furthermore, C contains no point interior to C_1.*

The first statement in (4) is easily verified. That the second holds is seen as follows. Any point of C is of the form

$$z = ax + (1 - a)y, \qquad x \in C_2, \ y \in C_1^-, \ a \in [0, 1].$$

Let x_1 be an inner point of C_1. From (3) it follows that on the segment $\overline{xx_1}$ there is a boundary point x_1' of C_1. Now by (1) C_1 has a supporting plane H_1 passing through x_1'. Since the point x_1' of H_1 is on the segment $\overline{x_1x}$, the points x_1 and x lie in opposite closed half-spaces of H_1. Since x_1 is interior to C_1, x_1 lies in the half-space of H_1 containing C_1. Hence the set C_1 and the point x lie in opposite closed half-spaces of H_1; that is, C_1 and x are separated by H_1. By (2) $\theta \in H_1$, so that C_1 and y are separated by H_1. Hence H_1 separates the sets C_1 and $\overline{xy}$, and the point z, which is contained in $\overline{xy}$, cannot be interior to C_1.

From (4) and (1) it now follows that C has a plane of support H passing through the point θ. But H is then a plane of support of C_2. Likewise it is a plane of support of C_1^- and hence of C_1. Since C_1^- and C_2 are on the same side of H, C_1 and C_2 are separated by H. Therefore K_1 and K_2 are separated by H.

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