

SOME THEOREMS ON SUBSERIES

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1. **Absolutely convergent series.** A simple calculation reveals that the arithmetic mean value of all subsums (including the void sum) of a given finite sum $s_n = u_1 + u_2 + \cdots + u_n$ is equal to $s_n/2$. In this section we shall show (see Theorem 1 below) that an integral mean value can be found, consistent with the preceding, for the sums of all infinite subseries of a given absolutely convergent series $\sum u_k = s$. We begin by defining a one-to-one correspondence between the set of all infinite subseries of a given absolutely convergent real series $\sum u_k = s$, and the set of all points on the interval $I \equiv (0 < \xi \leq 1)$. If ξ is any point of I then ξ admits a unique *nonterminating* binary representation of the form

$$(1.1) \quad \xi = 0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_k \cdots$$

where

$$(1.2) \quad \alpha_{k_i} = 1 \quad (1 \leq k_i < k_{i+1}; i = 1, 2, 3, \cdots); \quad \alpha_k = 0 \quad \text{otherwise.}$$

To the point ξ shall correspond the infinite subseries $\sum_i u_{k_i}$. Conversely, if $\sum_i u_{k_i}$ ($1 \leq k_i < k_{i+1}$) is a given infinite subseries of $\sum u_k$, we shall place it in correspondence with the point ξ of I defined by (1.1) and (1.2).

We now define a function $\phi(\xi)$ by setting $\phi(0) \equiv 0$ and

$$(1.3) \quad \phi(\xi) \equiv \sum_{k=1}^{\infty} \alpha_k u_k, \quad 0 < \xi \leq 1,$$

where $0.\alpha_1\alpha_2\alpha_3 \cdots \alpha_k \cdots$ is the nonterminating binary representation of ξ . In view of the above correspondence the set of all functional values $\phi(\xi)$ for ξ on I is evidently identical with the set of the sums of all infinite subseries of $\sum u_k$. This fact leads us to investigate the integrability of the function $\phi(\xi)$ and we find that the following lemma holds.

LEMMA 1. *The integral*

$$(1.4) \quad \int_0^1 \phi(\xi) d\xi$$

exists in the sense of Riemann, and has the value $s/2$.

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Before proceeding with the proof we may observe that this lemma furnishes a generalization of the fact mentioned above for finite sums. We may therefore express Lemma 1 in the form of the following theorem.

THEOREM 1. *In the sense of the integral (1.4), the mean value of the sums of all infinite subseries of a given absolutely convergent series is equal to one-half the sum of the series.*

PROOF OF LEMMA 1. We introduce the partial sums of the series in (1.3) and thus form a sequence of functions $\{\phi_n(\xi)\}$ for $n = 1, 2, 3, \dots$ where $\phi_n(0) \equiv 0$ and

$$(1.5) \quad \phi_n(\xi) \equiv \sum_{k=1}^n \alpha_k u_k, \quad 0 < \xi \leq 1.$$

For each fixed n it is possible to choose the set of digits $\alpha_1, \alpha_2, \dots, \alpha_n$ in 2^n distinct ways. We denote these choices by $\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{ni}$ ($i = 1, 2, 3, \dots, 2^n$). Then for each fixed i the set of all numbers ξ of the form $0.\alpha_{1i}\alpha_{2i} \dots \alpha_{ni}\alpha_{n+1}\alpha_{n+2} \dots \alpha_k \dots$ (nonterminating) comprises the interval $I_{ni} \equiv (0.\alpha_{1i}\alpha_{2i} \dots \alpha_{ni} < \xi \leq 0.\alpha_{1i}\alpha_{2i} \dots \alpha_{ni}111 \dots)$ of length 2^{-n} . The intervals I_{ni} ($i = 1, 2, \dots, 2^n$) are mutually disjoint and collectively exhaust the interval I . On the interval I_{ni} the function $\phi_n(\xi)$ has the constant value $\sum_{k=1}^n \alpha_{ki} u_k$. Therefore $\phi_n(\xi)$ is a step function.

Since $|\phi_n(\xi)| \leq \sum |u_k|$ and $|\phi(\xi) - \phi_n(\xi)| \leq \sum_{k>n} |u_k|$ for all $n = 1, 2, 3, \dots$ and all ξ ($0 \leq \xi \leq 1$), it follows that $\phi(\xi)$ is the uniform limit of a uniformly bounded sequence of step functions. This implies that (1.4) exists as a Riemann integral. To find its value we notice that $\phi(\xi) + \phi(1 - \xi) = \phi(1) = s$ for all values of ξ except those in the denumerable set T composed of all points having the form $k \cdot 2^{-n}$ for $k = 0, 1, 2, \dots, 2^n; n = 1, 2, 3, \dots$. If we denote by S the set $I - T$, we have in the sense of Lebesgue,

$$\int_S \phi(\xi) d\xi + \int_S \phi(1 - \xi) d\xi = s.$$

Since each integral on the left has the same value as the integral (1.4), the proof is complete.

By considering the series of real and imaginary parts it is easily seen that Lemma 1, and hence Theorem 1, remains valid for absolutely convergent series of complex terms.

In addition to the properties of the function $\phi(\xi)$ already mentioned we may in passing call attention to some further properties

that it possesses. In the first place it is apparent that each of the step functions $\phi_n(\xi)$ is continuous everywhere in the interval $0 \leq \xi \leq 1$ except perhaps at points of T , and that each is continuous on the left everywhere in this interval. On account of the uniform convergence of $\{\phi_n(\xi)\}$ to $\phi(\xi)$ it is clear that $\phi(\xi)$ possesses the same properties. Moreover, it is not difficult to see that at each point of T , say $\xi_0 = 0.\alpha_1\alpha_2 \cdots \alpha_n 0111 \cdots$, the saltus $|\phi(\xi_0) - \lim_{\xi \rightarrow \xi_0+} \phi(\xi)|$ is equal to $|u_n - \sum_{k>n} u_k|$. It follows at once that $\phi(\xi)$ is continuous everywhere if and only if u_k is of the form $a \cdot 2^{-k}$ ($k = 1, 2, 3, \cdots$), in which case $\phi(\xi) = a\xi$. Finally, by means of the property $\phi(\xi) + \phi(1 - \xi) = s$ for ξ in S , we can easily establish the equation

$$\frac{1}{2\delta} \int_{1/2-\delta}^{1/2+\delta} \phi(\xi) d\xi = \frac{s}{2}, \quad 0 < \delta \leq 1/2.$$

This shows that the mean value of the function $\phi(\xi)$ is $s/2$ in every subinterval of $(0, 1)$ whose midpoint is $1/2$.

2. Conditionally convergent series. Throughout this section $\sum u_k$ will denote a conditionally convergent real series. For series of this type the corresponding sequence of functions $\{\phi_n(\xi)\}$ defined by (1.5) will again be a sequence of step functions. Moreover, if $\sum_{i \in k_i} u_{k_i}$ ($k_i < k_{i+1}$) denotes formally a given infinite subseries of $\sum u_k$ it is clear that the behavior as $n \rightarrow \infty$ of the sequence $\{\sum_{i=1}^n u_{k_i}\}$ is identical with the behavior of the corresponding sequence $\{\phi_n(\xi)\}$, where ξ is defined by (1.1) and (1.2). In studying the character of subseries we may therefore confine our attention to the sequence $\{\phi_n(\xi)\}$.

An interesting subset of I is the set G of all points ξ which correspond to convergent subseries of $\sum u_k$. We shall prove that G is a set of the first category by establishing the following stronger result.

THEOREM 2. *For all points ξ of I except those in a set H of the first category we have*

$$(2.1) \quad \liminf_n \phi_n(\xi) = -\infty, \quad \limsup_n \phi_n(\xi) = +\infty.$$

PROOF. We recall the sets T and S as defined above in the proof of Lemma 1. We may regard S itself as a metric space S^* by conserving the euclidean notion of distance. It is clear that S^* is of the second category on itself. Furthermore, since all points of discontinuity for the step function $\phi_n(\xi)$ are included in the set T it follows that each of these functions is continuous on S^* .

Let A denote the subset of S^* of all points ξ for which $\limsup_n \phi_n(\xi) < \infty$. The points of A correspond to all subseries whose partial sums

are bounded from above, except those in correspondence with points of T .

The set A is of the first category on S^* . To establish this fact let A_m ($m=1, 2, 3, \dots$) be that subset of A of all points ξ such that $\phi_n(\xi) \leq m$ for all $n=1, 2, 3, \dots$. Then $A = \sum_{m=1}^{\infty} A_m$, and each of the sets A_m is closed in S^* since $\phi_n(\xi)$ is continuous in S^* . To obtain a contradiction we assume that A is of the second category. Then at least one of the sets A_m , say A_μ , must be such that its closure, namely A_μ itself, contains all points of S^* which lie in a certain subinterval of I . We may assume this subinterval to be of the form

$$(2.2) \quad 0.\beta_1\beta_2 \cdots \beta_p < \xi < 0.\beta_1\beta_2 \cdots \beta_p + 2^{-p},$$

in the binary scale. We now define the point $\xi_1 \equiv 0.\beta_1\beta_2 \cdots \beta_p\gamma_{p+1}\gamma_{p+2} \cdots \gamma_k \cdots$ by setting $\gamma_k = 1$ if $u_k \geq 0$ and $\gamma_k = 0$ if $u_k < 0$ ($k = p+1, p+2, \dots$). By a familiar property of conditionally convergent series, infinitely many of the γ_k are 0 and infinitely many are 1. Since the nonterminating representation of each point in T is ultimately comprised wholly of 1's, it is clear that ξ_1 belongs to S^* and moreover, lies in the interval (2.2). On the other hand we have $\lim_n \phi_n(\xi_1) = +\infty$, since the subseries corresponding to ξ_1 is divergent to $+\infty$. It follows that ξ_1 cannot belong to A_μ , and this contradiction completes the argument.

From the fact just established we conclude that A is likewise of the first category on I .

We now let B denote the subset of S^* of all points ξ for which $\liminf_n \phi_n(\xi) > -\infty$. The set B evidently coincides with the set A defined with respect to the series $\sum(-u_k)$. Thus B is also of the first category on I .

Finally, the set T , being denumerable, is of the first category on I , and we observe that at each point of T the sequence $\{\phi_n(\xi)\}$ is convergent.

It follows that $H \equiv A + B + T$ is a set of the first category, and that all points of I for which at least one of the relations (2.1) fails to hold are contained in the set H . This completes the proof. As a concluding remark we may observe that the set G , defined above, is of the first category since it is a subset of H .

Theorem 2 was suggested by a theorem of the same general type established recently by Agnew² in connection with rearrangements of conditionally convergent series. The domain space I of Theorem 2 occupies the role played by Agnew's metric space E in which a point $x \equiv \{k_i\}$ is a rearrangement of the sequence $(1, 2, 3, \dots)$ of positive

² Agnew, *On rearrangements of series*, this Bulletin, vol. 46 (1940), pp. 797-799.

integers and the distance (x, y) between the points $x \equiv \{k_i\}$ and $y \equiv \{h_i\}$ of E is given by the formula of Fréchet

$$(2.3) \quad (x, y) \equiv \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|k_i - h_i|}{1 + |k_i - h_i|}.$$

The analogous approach for subseries of conditionally convergent series may be employed to yield a further theorem (see Theorem 3 below) of the same nature as Theorem 2. To this end we denote by D the metric space in which a point x is a strictly increasing infinite sequence $\{k_i\}$ of positive integers and the distance (x, y) between the points x and $y \equiv \{h_i\}$ is given by (2.3).

Unlike the space E of Agnew, the space D is complete, and therefore, by the Baire theorem, of the second category. The proof of completeness is entirely straightforward and may be left to the reader. It is likewise a simple matter to verify that the sequence $\{x_n\} \subset D$ converges to $x_0 \in D$, where $x_n \equiv \{k_i^n\}$ for $n = 0, 1, 2, \dots$, if and only if integers N_i exist such that $k_i^n = k_i^0$ for all $n > N_i$ ($i = 1, 2, 3, \dots$). We shall use this fact presently.

To each x in D there corresponds an infinite subseries $\sum_i u_{k_i}$ of $\sum u_k$, and, of course, conversely. If for each x in D we set

$$(2.4) \quad f_n(x) \equiv \sum_{i=1}^n u_{k_i}, \quad n = 1, 2, 3, \dots$$

we may then state the following analogue of Theorem 2.

THEOREM 3. *For all points x of D except those in a set W of the first category we have*

$$(2.5) \quad \liminf_n f_n(x) = -\infty, \quad \limsup_n f_n(x) = +\infty.$$

PROOF. Let U denote the set of all x in D for which $\limsup_n f_n(x) < \infty$, and let U_m ($m = 1, 2, 3, \dots$) denote the subset of U on which $f_n(x) \leq m$ for all $n = 1, 2, 3, \dots$. Then $U = \sum_{m=1}^{\infty} U_m$. Moreover, each of the sets U_m is closed. For let $x_0 \equiv \{k_i^0\}$ be any point of the derived set U'_m , and let $\{x_p\}$ be an arbitrary sequence in U_m converging to x_0 . If $x_p \equiv \{k_i^p\}$ for $p = 1, 2, 3, \dots$, then, by the remark made above, there exist integers N_i such that $k_i^p = k_i^0$ for all $p > N_i$ ($i = 1, 2, 3, \dots$). If we let $P_n \equiv \max(N_1, N_2, \dots, N_n)$, then $f_n(x_p) = f_n(x_0)$ for all $p > P_n$. Since $f_n(x_p) \leq m$ ($n, p = 1, 2, 3, \dots$) it follows that $f_n(x_0) \leq m$ ($n = 1, 2, 3, \dots$). Thus x_0 belongs to U_m and U_m is therefore closed.

If we assume that U is of the second category at least one of the closed sets U_m , say U_μ , must contain a sphere $K \equiv [(x, x_0) \leq r]$ of posi-

tive radius r . Let the center x_0 be the sequence $\{k_i^0\}$, and let s be chosen so large that $2^{-s-1} + 2^{-s-2} + \dots < r$. Now a point $X \equiv \{j_i\}$ of D exists such that $\lim_n f_n(X) = +\infty$, and such that $j_{s+1} > k_s^0$. If we define x_1 as $(k_1^0, k_2^0, \dots, k_s^0, j_{s+1}, j_{s+2}, \dots)$, then x_1 belongs to K and $\lim_n f_n(x_1) = +\infty$. Consequently x_1 cannot be a point of U_μ and this contradiction establishes U as a set of the first category.

In a similar fashion it may be shown that the set V of all x in D for which $\liminf_n f_n(x) > -\infty$ is likewise a set of the first category. Hence if we set $W \equiv U + V$ the theorem follows.

Finally, let $\sum u_k$ be a convergent series of complex terms for which $\sum |u_k| = +\infty$, and for this series let $\phi_n(\xi) [f_n(x)]$ be defined as in (1.5) [(2.4)]. We may consider the series of real and imaginary parts in the light of Theorem 2 [Theorem 3] and thus show that the set of all ξ on I [x in D] for which we have $\limsup_n |\phi_n(\xi)| < \infty$ [$\limsup_n |f_n(x)| < \infty$] is a set of the first category.

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A FORMULA FOR THE DIRECT PRODUCT OF CROSSED PRODUCT ALGEBRAS

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1. Introduction. In this note we wish to present a uniform treatment of certain properties of crossed products. A crossed product over any field F is an algebra determined by a finite, separable, normal extension N of F , with a Galois group Γ , and a certain factor set¹ h of elements $h_{S,T}$ in N , for automorphisms S and T in Γ . The crossed product (N, Γ, h) consists of all sums $\sum u_S z_S$, where the coefficients z_S lie in N , and the fixed elements u_S have the multiplication table

$$(1) \quad u_{SUT} = u_{ST} h_{S,T}, \quad zu_S = u_S z^S, \quad z \text{ in } N.$$

Let K be a normal subfield of N , corresponding to the subgroup Δ of the Galois group Γ . A factor set g in N is called *symmetric* in Δ if $g_{S,T} = g_{U,V}$ whenever SU^{-1} and TV^{-1} are in Δ .

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¹ Definitions are given in A. A. Albert, *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939. Theorems cited below without explicit source all refer to this work.