25. L. R. Wilcox: *Extensions of semi-modular lattices.* III.

The author's result (abstract 47-5-208) is extended to all complemented semi-modular lattices of dimension equal to or greater than 4. The following theorem is proved. Let L be left complemented (abstract 47-9-356) with the further property that $b, c \in L$, $bc \neq 0$ implies $(a+b)c = a+bc$ for $a \leqq c$; suppose also that there exists in L a chain of length 6. Then there exists a complemented modular lattice Λ containing L order-isomorphically and having the properties (a) $a \in L$, $a \neq 0$, $b \geqq a$ implies $b \in L$, and (b) for $a \in \Lambda$, $b \in L$, $a \leqq b$ there exists $c \in L$ such that c is a complement (in Λ) of a in b. Properties (a), (b) characterize Λ uniquely up to isomorphisms. This theorem is a lattice-theoretic generalization of well known imbeddings of affine and hyperbolic spaces into projective spaces. (Received November 24, 1941.)

26. Leonard Carlitz: *q-Bernoulli numbers and polynomials.*

Rational functions of q are defined by means of $q(qb+1)^m = b^m$ $(m>1)$, where after expansion b^m is replaced by b_m; "polynomials" are defined by $b_m(x) = \sum_{\alpha=0}^{m} C_{m,\alpha} q^{\alpha x}[x]^{m-\alpha} b_\alpha$, where $[x] = (q^x - 1)/(q-1)$. Many of the properties of the ordinary Bernoulli numbers and polynomials are readily extended to these quantities; in addition there are certain formulas in the generalized case that are not easily specialized to the case $q=1$. Among possible explicit formulas for b_m may be mentioned $b_m = \sum_{s=0}^{m} 1/[s+1] \sum_{\alpha=0}^{s} (-1)^\alpha \left[{s \atop \alpha}\right] q^{\alpha(\alpha+1-2s)/2} \cdot [\alpha]^m$, which leads at once to a generalized Staudt-Clausen theorem: $b_m = \sum_{s=2}^{m+1} N_s(q)/F_s(q)$ $(m>0)$, where $F_s(q)$ is the cyclotomic polynomial and $\deg N_s < \deg F_s$. (Received November 24, 1941.)

27. Joseph Lehner: *The Ramanujan identities and congruences for powers of eleven.* Preliminary report.

The author proves the existence of a "Ramanujan identity" for the modulus 11^α $(\alpha \geqq 1)$. For $\alpha = 1, 2$, this identity implies the Ramanujan conjecture: $p(n) \equiv 0 \pmod{11^\alpha}$ if $24n \equiv 1 \pmod{11^\alpha}$. The methods used are those of Rademacher's paper *The Ramanujan identities under modular substitution* (to be published in the American Journal of Mathematics). A modification of Hecke's T-operator is used. This operator is defined as follows: $U_{11}F(\tau) = \Sigma_\lambda F(\tau + \lambda/11)$, $\lambda \bmod 11$. If $F(\tau)$ is a modular function belonging to $\Gamma_0(121)$, that subgroup of the modular group defined by $c \equiv 0 \pmod{121}$, then $U_{11}F$ belongs to $\Gamma_0(11)$. Then it can be expressed as a polynomial in $A(\tau)$, $B(\tau)$, certain well known functions which constitute a basis for $\Gamma_0(11)$. By taking F to be $\eta(121\tau)/\eta(\tau)$, where $\eta(\tau)$ is the well known elliptic modular function of Dedekind, we obtain the desired Ramanujan identity for the modulus 11. Identities for higher powers of 11 are then obtained by a two-fold induction, one for even α, the other for odd α. The possibility of proving Ramanujan's conjecture for higher values of α $(\alpha > 2)$ is being investigated. (Received November 19, 1941.)

Analysis

28. C. B. Barker: *The Lagrange multiplier rule for two dependent and two independent variables.*

Let $\bar{z}_1(x, y)$ and $\bar{z}_2(x, y)$ be of class C'''' on a closed simply connected region $\overline{G}$ of class C_α''' and minimize (1) $\iint_G f(x, y, z_1, z_2, p_1, p_2, q_1, q_2)dxdy$ among all pairs of functions $z_1(x, y)$ and $z_2(x, y)$ which coincide on the boundary G^* with $\bar{z}_1(x, y)$ and $\bar{z}_2(x, y)$, respectively, and which satisfy (2) $\phi(x, y, z_1, z_2, p_1, p_2, q_1, q_2) = 0$ on G; assume that f

and ϕ are of class C'''' in their arguments everywhere. It is proved in this paper that there exists a unique multiplier $\lambda(x, y)$ of class C' on $\overline{G}$ such that $\bar{z}_1$ and $\bar{z}_2$ satisfy the equations $(\partial/\partial x)(f_{p_i}-\lambda\phi_{p_i})+(\partial/\partial y)(f_{q_i}-\lambda\phi_{q_i})=f_{z_i}-\lambda\phi_{z_i}$, $i=1, 2$, provided that the pair $(\bar{z}_1, \bar{z}_2)$ is "quasi-normal" with respect to the equation (2). The quasi-normality requires (i) that $\phi_{p_1}\phi_{q_2}-\phi_{p_2}\phi_{q_1}\neq 0$ on $\overline{G}$ for the pair $(\bar{z}_1, \bar{z}_2)$, (ii) that to any pair of functions ζ_1, ζ_2 which vanish on G^* and satisfy the equation of variation on G shall correspond a 1-parameter family of solutions $z_i(x, y; \mu)$ of (2) which coincide with $\bar{z}_i(x, y)$ on G^* for each μ near zero, which reduce to $\bar{z}_i$ for $\mu=0$, and which have the ζ_i as their variations for $\mu=0$, and (iii) that a certain other rather complicated differential expression not be an exact differential on any rectangle on G. (Received November 3, 1941.)

29. E. F. Beckenbach: *Painlevé's theorem and the analytic prolongation of a minimal surface.*

The following generalization of Painlevé's theorem is obtained. If $x_j(u, v)$, $x_j'(u, v)$, $j=1, 2, 3$, are defined, respectively, in contiguous domains D, D' having a rectifiable arc C of boundary in common; if $x_j(u, v)$, $x_j'(u, v)$ are harmonic and satisfy $E=G$, $F=0$ in their respective domains of definition; if $X_j(u, v)$, $X_j'(u, v)$ denote the direction-cosines of the normals to the minimal surfaces S, S' thus determined; and if the above functions remain continuous on C and satisfy the relations $x_j(u, v)=x_j'(u, v)$, $X_j(u, v)=X_j'(u, v)$, $j=1, 2, 3$, for all points on C; then S, S' are analytic prolongations of each other across C. (Received November 22, 1941.)

30. E. F. Beckenbach and Maxwell Reade: *Mean-values and harmonic polynomials.*

It is shown that a necessary and sufficient condition that, for a fixed $n\geqq 3$, a function $f(x, y)$, superficially summable in a finite domain D, assume at each point P of D a value equal to its areal (or peripheral) mean on every regular n-gon whose center is at P and whose circumscribed circle lies together with its interior in D, is that $f(x, y)$ be a harmonic polynomial of degree at most $n-1$. The familiar characteristic property of harmonic functions in terms of circular mean-values appears as a limiting case. (Received November 22, 1941.)

31. Stefan Bergman: *On operators in the theory of partial differential equations and their application.*

There exists for every equation $L(U)\equiv\Delta U+C(z, \bar{z})U=0$, $z=x+iy$, $\bar{z}=x-iy$, a function $E(z, \bar{z}, t)$ such that every solution U, of $L(U)=0$ can be represented in the form $U=P(f)=\int_{-1}^{+1}Ef(\zeta)d\tau$, $\zeta=z(1-t^2)/2$, $d\tau=(1-t^2)^{-1/2}dt$; f being a suitable analytic function of one complex variable ζ. This representation is applied to the solution of boundary value problems. Let $\phi(z, \bar{z})=c$ be a curve k in the xy-plane. If $E(z, \bar{z}, t)=E^*(z, z+\phi(z, \bar{z}), t)$ then $P(f)$ and $H(f)=\int_{-1}^{+1}E^*f(\zeta)d\tau$ assume on k the same values. Thus: if F is the analytic function which assumes on k the given values, V, and $f=h$, is the solution of $H(f)=F$, $U=P(h)$ will be the solution of $L(U)=0$ assuming on k the values V. Using the representation $U=\int_0^{\pi}\int_{-1}^{+1}E(r, t, v)f(\zeta)d\tau dv$ valid for the solutions of $T(U)\equiv U(x, y, z)+C(r^2)U=0$, $r^2=x^2+y^2+z^2$, an analogous method for the solution of boundary value problems of $T(U)=0$ is obtained. Using the last representation the following residue theorem is proved: there exist to every function U_1 two functions U_2 and U_3, $T(U_k)=0$, such that $\int_C(U_1dx+U_2dy+U_3dz)=0$ if C lies on a sphere with the center at origin and can be reduced to a point in the regularity domain of U_k. (Received November 24, 1941.)

32. Salomon Bochner and W. T. Martin: *A class of removable singularities in several complex variables.*

Given a function $f(z_1, \cdots, z_n)$ continuous in a region R and analytic in R except possibly on some exceptional set E in R. Under what conditions on the set E does this imply that f is analytic throughout R? This question is investigated. One sufficient condition is found to be that (measure $E_\epsilon)/\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ where E_ϵ is the set of points in R whose distance from E is less than ϵ. (Received November 24, 1941.)

33. J. W. Bradshaw: *On a certain class of continued fractions.*

Continued fractions which are equivalent to the series $S_k = \sum n^{-k}$ for several values of k are generalized by the addition of a variable term to each of the partial denominators. Series equivalent to the resulting continued fractions are set up by means of two systems of difference equations. Application is made to the calculation of the remainder of S_k after summing a few terms. Some of the continued fractions considered are very rapidly convergent. For example, it has been possible, by means of one of them, to recompute Stieltjes' thirty-place value of S_3. (Received November 21, 1941.)

34. J. L. Doob: *Topics in the theory of Markoff chains.*

Let the matrix function $p_{ij}(t)$ $(t>0)$ $i, j=1, 2, \cdots$ satisfy the following conditions: $p_{ij}(t) \geqq 0$, $\sum_j p_{ij}(t) = 1$, $p_{ik}(s+t) = \sum_j p_{ij}(s) p_{jk}(t)$. Then the $p_{ij}(t)$ can be interpreted as the transition probabilities of a stochastic (Markoff) process. The properties of the $p_{ij}(t)$ (especially as $t \rightarrow \infty$ and $t \rightarrow 0$) and the relations between the Markoff process and the matrix function are examined in detail. For example, necessary and sufficient conditions that the Fokker-Planck differential equations in the $p_{ij}(t)$ hold are formulated in terms of the properties of the state function x_t (taking on the value j if the jth state is assumed at time t). (Received October 30, 1941.)

35. H. J. Ettlinger: *The theory of the Riesz integral.*

In this paper the author defines the definite integral following Riesz by means of step functions and, making use of a simple set of axioms, obtains a characterization of the Riemann and the Lebesgue integrals. Other more general integrals are discussed. (Received November 24, 1941.)

36. Abe Gelbart: *On functions of two complex variables with bounded real parts.*

Let $f(z_1, z_2)$ be regular in the interior of a finite four-dimensional domain D^4, belonging to the class $\mathfrak{D}^4$ of domains equivalent to the bicylinder, and having a bounded real part in D. The author obtains an upper bound for $f(z_1, z_2)$ in terms of $A = \max_{\{z_1 z_2\} \in D} \text{Re} f(z_1, z_2)$, $f(0, 0)$ and the domain only. The upper bound becomes unbounded only when the point $\{z_1, z_2\}$ approaches a two-dimensional surface on the three-dimensional boundary, so that a consequence of the inequality is that a function regular in the interior of the domain D^4 with a bounded real part is regular almost everywhere on the three-dimensional boundary. From this result, an upper bound for $|\partial^{m+n} f(z_1, z_2)/\partial z_1^m \partial z_2^n|$ is obtained again in terms of A, $f(0, 0)$, and the domain only, for the particular domain bounded by the analytic hypersurfaces $S_1^3 = E[z_1 = re^{i\lambda_1}, 0 \leqq \lambda_1 \leqq 2\pi]$ and $S_2^3 = E[z_1 = re^{i\lambda_2} + p(re^{i\lambda_2}) z_2]$. (Received November 24, 1941).

37. Einar Hille: *On the oscillation of differential transforms. II.*

G. Pólya and N. Wiener have shown that $V[(D^2-\lambda)f(x)] \geqq V[f(x)]$ where $f(x+2\pi)=f(x)$, $V[g(x)]$ is the number of sign changes of $g(x)$ in $(0, 2\pi)$, $D=d/dx$, and $\lambda>0$. From this point of departure they proved that if $V[D^{2n}f(x)] \leqq N$ for all n, then $f(x)$ is a trigonometric polynomial of degree not greater than $N/2$, as well as other results relating $V[D^{2n}f(x)]$ with the analytical properties of $f(x)$. In the present paper this theory is extended to a large class of linear second order differential operators ϑ. For such operators $V[(\vartheta-\lambda)f(x)] \geqq V[f(x)]$, $\lambda>0$, if $f(x)$ satisfies the appropriate boundary conditions. Further, $\liminf_{n\to\infty} V[\vartheta^n f(x)] < \infty$ implies that $f(x)$ is a finite sum of characteristic functions of a corresponding boundary problem. Included in the discussion are the differential operators associated with the names of Bessel, Hermite, Jacobi, Laguerre, Legendre, and Mathieu as well as those of the classical boundary value problems of the Sturm-Liouville type with analytical coefficients. For the Jacobi and Legendre operators partly sharper results have been proved by other methods by G. Szegö in the first paper in this series. (Received November 22, 1941.)

38. Mark Kac: *On convergence of certain series of functions.*

Let $\phi(x)$ be such that (1) $\phi(x+2\pi)=\phi(x)$, (2) $|\phi(x_1)-\phi(x_2)| \leqq M|x_1-x_2|^\alpha$ $(0<\alpha\leqq 1)$, (3) $\int_0^{2\pi}\phi(x)dx=0$, and let $\{n_k\}$ be a sequence of integers satisfying the gap condition $n_{k+1}/n_k>q>1$ $(k=1, 2, \cdots)$. The following theorem is then proved: If $\sum_{k=1}^{\infty}a_k^2<\infty$, the series $\sum_{k=1}^{\infty}a_k\phi(n_kx)$ converges almost everywhere. (Received November 21, 1941.)

39. R. B. Kershner: *The continuity of functions of many variables.*

It is known that a function of n real variables may be continuous with respect to each of the variables separately in a given region and still have discontinuities. In fact the set of these discontinuities may have full measure and may include all points of an $(n-2)$-dimensional region. In this paper it is shown that the set of discontinuities cannot have Menger dimension greater than $(n-2)$. The principal tool is Baire's theorem and correspondingly a stronger but more involved statement is proved involving the category of the discontinuities. With the assumption of a very little extra smoothness, much less than a Lipschitz condition with respect to the separate variables, it is shown that the discontinuities are nowhere dense. Examples are given incating to what extent the results given are complete. (Received November 25, 1941.)

40. H. N. Laden: *An interpolation polynomial involving derivatives of a prescribed function.*

In an earlier paper (Duke Mathematical Journal, vol. 8 (1941)), the author considered an interpolation polynomial $F_n(x)$ of degree less than or equal to $4n-1$ which takes preassigned values $y_1, y_2, \cdots, y_n$ at prescribed abscissas $x_1, x_2, \cdots, x_n$ and such that $F_n^{(\nu)}(x_i)=0$ $(i=1, 2, \cdots, n;\ \nu=1, 2, 3)$, especially where the abscissas are zeros of classical orthogonal polynomials and $y_i=f(x_i)$ $(i=1, 2, \cdots, n)$, $f(x)$ being an arbitrary continuous function. Earlier conjectures, that some convergence theorems for Laguerre abscissas are "best possible" are shown to be only partially correct. Also, $F_n(0)$ is investigated for Laguerre abscissas, $F_n(\pm 1)$ for Jacobi abscissas. It is shown, further, that for Newton (equidistant) abscissas on $(-1, 1)$, there exist continuous functions on $[-1, 1]$ such that the corresponding $F_n(x)$ does not converge uniformly to $f(x)$ on $[-1, 1]$ as $n\to\infty$. In addition, the case where $F_n^{(\nu)}(x_i)=y_i$ $(\nu=1, 2, 3;$ $i=1, 2, \cdots, n)$ are preassigned not necessarily zero is considered for Jacobi abscissas and for functions with prescribed moduli of continuity. The methods used are due to Fêjer, Shohat and Szegö. (Received November 24, 1941.)

41. E. J. McShane: *On Perron integration.*

Perron's definition of integral has the advantage of elegance, and yields very simple proofs of many theorems; but the most general theorem on integration by parts has not been proved for the Perron integral except by the detour of proving the equivalence of the Perron integral and the special Denjoy integral. Here a definition of integral is presented which is shown to be equivalent to that of Perron, but has the advantage of permitting a direct proof of the general theorem on integration by parts. The new definition is closely related in form to that of Perron; the distinction is that Perron's major functions are replaced by pairs (right majors and left majors), and analogously for minor functions. (Received November 24, 1941.)

42. E. J. McShane: *The derivative of the indefinite Lebesgue integral.*

This note contains a proof of the fundamental theorem on the derivative of the indefinite Lebesgue integral which (it is hoped) has the advantages of simplicity and of requiring little preparatory material. (Received November 24, 1941.)

43. P. T. Maker: *The Cauchy theorem for functions on closed sets.*

Let $f(z)$ be defined on E, a bounded, closed set in the complex plane and such that every point z is the limit of two sequences of points lying on curves having noncollinear tangents at z. It is shown that if $f(z)$ has a derivative at each point of E, there is a sequence of coverings $\{R_n\}$ of E, and a continuation, $f^*(z)$, of $f(z)$ to the rest of the plane, for which $\lim_{n\to\infty}\int_{R_n}f^*(z)dz=0$ and $\lim_{n\to\infty}mR_n=mE$. Conditions weaker than the existence of the derivative are found which give the same result. (Received November 25, 1941.)

44. E. J. Mickle: *Associated double integral variation problems.*

By generalizing the concept of adjoint minimal surfaces, Haar (Mathematische Annalen, vol. 100 (1928), pp. 481–502) has shown that with every extremal surface of a double integral variation problem $J[z]$ in which the integrand function is of the form $F(p, q)$, there can be associated an adjoint extremal surface which is itself an extremal surface of an associated adjoint variation problem. The results of Haar can be extended so that with every extremal surface of the problem $J[z]$ there can be associated twenty-four surfaces such that these surfaces are themselves extremal surfaces, respectively, of twenty-four associated double integral variation problems. The transformations defining the associated problems form a group of order twenty-four. Analogous results can be obtained for parametric double integral variation problems. (Received November 18, 1941.)

45. G. C. Munro: *Systems of linear differential equations with constant coefficients.*

This paper presents a new, simple and direct treatment of systems of linear homogeneous differential equations with constant coefficients. (Received October 9, 1941.)

46. S. B. Myers: *An existence theorem for a self-adjoint system of second-order, linear, homogeneous differential equations.*

In 1929 Morse (Mathematische Annalen, vol. 103 (1930), p. 66) generalized the classical Sturm comparison theorem to n dimensions. One consequence of his results is that if $p_{ij}(x)\eta_i\eta_j \geqq A^2\sum \eta_i^2$ and $r_{ij}(x)\eta_i\eta_j \leqq B^2\sum \eta_i^2$ for all $(\eta) = (\eta_1, \cdots, \eta_n)$, then there exists a solution $(y) \not\equiv (0)$ of the system of differential equations $d/dx(r_{ij}y_j') + p_{ij}y_i = 0$ vanishing at $x = a$ and vanishing later before or at $x = a + \pi B/A$. It is assumed that r_{ij} and p_{ij} are of class C' and $r_{ij}\eta_i\eta_j$ is positive definite. In the present paper the same result is proved under weaker hypotheses: namely, with the above inequalities replaced by $\sum p_{ii} \geqq nA^2$, $\sum r_{ii} \leqq nB^2$. This result has applications to a problem in differential geometry in the large (Myers, Duke Mathematical Journal, vol. 8 (1941), pp. 401–404). (Received November 21, 1941.)

47. N. M. Oboukhoff: *The historical development of total differential as the principal part of the increment of a function of several variables.*

There is more than formalism in the differential calculus of Leibniz, that has irrevocably been assimilated by mathematics. This is his basic concept of the differential: what Leibniz calls "difference" is actually "differential" in the sense of the principal part of the increment of a function; likewise in Leibniz' hands analysis became a particular system of calculus; Leibniz as logician and mathematician promoted the same form of structure, that of calculus. Also, he used pragmatico-operational approach where logical foundations were not strong enough. We can discern the general outlines of neoclassical analysis in the characteristics of Leibniz' doctrine. The first half of the eighteenth century showed deterioration of the foundations of calculus in contrast to an irresistible impetus of its growth and development of applications. Later, a balance was restored by Lagrange; the foundations of calculus were formulated by him in a kind of synthesis of the two major doctrines, those of Leibniz and of Newton. In the nineteenth century Cauchy, following in the footsteps of Leibniz and Lagrange proved that the total differential could be determined independently of derivatives as the principal part of the increment of a function, although he did not use this term; Weierstrass introduced it. (Received November 24, 1941.)

48. O. G. Owens: *An explicit formula for the solution of the ultrahyperbolic equation in four independent variables.*

By adapting H. Lewy's generalization of the Riemann integration method for a linear hyperbolic equation in two independent variables, an explicit formula is derived which gives the value of the solution of the ultrahyperbolic equation in four independent variables. The formula requires initial values which may be sufficiently differentiated, that is, regular initial values, and also certain Riemann functions. It is shown that there must necessarily be two integro-differential equations which hold for the solution and its normal derivative on the initial surface. These results are carried further in the case of a hyper-plane, where, by means of an example, the equalities are shown to be not identically satisfied. (Received October 22, 1941.)

49. J. F. Paydon and H. S. Wall: *An extension of the Stieltjes continued fraction theory.*

The authors show that if a_n is in the parabola $|z| - R(z) = h/2$, $0 < h \leqq 1$, and $\sum |b_n|$ diverges, $b_1 = 1$, $a_{n+1} = 1/b_nb_{n+1}$, $(n = 1, 2, 3, \cdots)$, then the continued fraction $1/1 + a_2t/1 + a_3t/1 + \cdots$ converges uniformly on the interior of the cardiod $|t|^2 = [|t| + R(t)]/2h$. If $\sum |b_n|$ converges the sequences of even and odd approximants converge uniformly to separate limits. The convergence theorem of Stieltjes (a_n real and positive) appears as the limiting case $h \to 0$. The theorem solves the problem of "limitar-periodic" continued fractions in important cases, for example, if c is not real and less than or equal to $-1/4$ it is easy to determine a circle with center c which bounds a convergence region for $1/1+a_2/1+a_3/1+\cdots$. Also included is a theorem recently announced by Leighton and Thron (abstract 47-7-308). (Received November 5, 1941.)

50. Edmund Pinney: *Theory of functions on linear topological spaces to Banach spaces.*

This paper deals for the most part with the theory of Banach valued functions defined on a linear topological space. The Riemann integral and the linear topological M-differential are defined, and various of their properties are investigated. Finally, these results are applied in the proof of existence theorems for the abstract differential equation system $df(x, v, u) = F(x, f, dx)$, (where differentiation is with respect to x only), $f(u, v, u) = v$; and for the partial differential equation $g_1(u, v; h) + g_2(u, v; F(u, v, h)) = 0$. When $F(x, y, z)$ is linear in y, it is shown that under the conditions given, $f(x, v, u)$ is linear in v, and differentiable with respect to all three places. (Received October 24, 1941.)

51. G. Y. Rainich: *Factorization of polynomials, in a ring, with application to partial differential equations.* Preliminary report.

Matrices have been used for the purpose of obtaining a system of first order equations equivalent to a partial differential equation of the second order (in the same sense that the Cauchy-Riemann equations are equivalent to the Laplace equation) by Brill, Dirac and others. The procedure consists in expressing a second order linear differential operator as a product of two first order operators with matrix coefficients. With a view of extending these results the present paper expresses a quadratic polynomial with indeterminates as variables as the product of a linear and *another quadratic* polynomial; the coefficient field must for this purpose be extended into a noncommutative ring. This ring is studied geometrically by considering its elements as operators on a vector space; upon introduction of coordinates these operators are expressed as matrices. As special cases we obtain the above mentioned solution of Dirac and another solution which leads to relations previously obtained by Duffin. The structure of the general solution is studied in terms of these two special solutions. A generalization in which the second factor is a polynomial of degree higher than the second is indicated. (Received November 24, 1941.)

52. Maxwell Reade: *Some remarks on subharmonic functions.* Preliminary report.

Continuous subharmonic functions are characterized by the property that they are dominated by their circular averages, or mean-values (T. Radó, *Subharmonic Functions*, Berlin, 1937, pp. 7–8). The author investigates the essentiality of the use of circular averages; he uses general regular n-gonal averages instead, for $n \geqq 3$. A typical result is the following. *If $f(x, y)$ is continuous in a bounded simply connected domain D, and if $f(x, y) \leqq 1/4h^2 \int_{-h}^{h}\int_{-h}^{h} f(x+\xi, y+\eta)\,d\xi\, d\eta$, holds for all h sufficiently small, for each point (x, y) of D, then $f(x, y)$ is subharmonic in D.* The converse does not hold, in general, without further hypotheses placed upon $f(x, y)$; a simple counterexample is the (sub)harmonic polynomial $x^4 - 6x^2y^2 + y^4$. However, *if $\Delta f(x, y)$ throughout D,* where Δ is the Laplacian operator, *then $f(x, y)$ is dominated by its square averages, for* *sufficiently small squares*. The squares used here are for illustrative purposes only; general n-gons are used in the paper. (Received November 17, 1941.)

53. G. E. Reves and Otto Szász: *Some theorems on double trigonometric series.*

The paper generalizes to two variables the theorems of Cantor-Lebesgue and of Fatou-Denjoy-Lusin on trigonometric series, and two theorems of Szász on absolute convergence of Fourier series. The mode of proof is an adaptation of the methods used in the one variable case. (Received November 10, 1941.)

54. R. M. Robinson: *Bounded univalent functions.*

Let $f(z)$ be regular and univalent for $|z|<1$, $|f(z)|<1$ there, and $f(0)=0$. Using the method of Löwner, a detailed study is made of the inequalities involving $|f'(0)|$, $|z_0|$, $|f(z_0)|$, and $|f'(z_0)|$. A typical result is that if $|f'(0)|$ and $|z_0|$ are given, the largest possible value of $|f'(z_0)|$ is attained for a mapping of $|z|<1$ on the unit circle with a radial slit provided $|z_0|\leqq 1/2$, but not in all cases. Unbounded univalent functions are considered as limits of bounded functions. Let $F(z)$ be regular and univalent for $|z|<1$, with $F(0)=0$ and $F'(0)=1$. Relations between $|z_0|$, $|F(z_0)|$, and $|F'(z_0)|$ are studied. In particular, sharp bounds for $|F'(z_0)|$ in terms of $|F(z_0)|$ are given. A striking result is that if $|F(z_0)|\leqq 1/4$ then $|F'(z_0)|<2.07$, but that if $|F(z_0)|$ has a given value greater than 1/4, no upper bound for $|F'(z_0)|$ exists. (Received October 21, 1941.)

55. H. M. Schwartz: *On sequences of Stieltjes integrals.*

This paper continues the discussion of convergence criteria for sequences of Riemann-Stieltjes integrals $J_n=\int_a^b f dg_n$ of a former paper (*Sequences of Stieltjes integrals*, this Bulletin, vol. 47 (1941), pp. 947–955); it contains a development of sets NS of necessary and sufficient conditions for the convergence of $\{J_n\}$ for different subclasses of the class F of functions f for which J_n exist (g_n being assumed to be of bounded variation), and it concludes with an indication of the possibility of applying the results to certain linear operations. From results obtained in the aforesaid paper one can conclude that for the class D of functions of F which have no singularities of the second kind, a NS set is given by the requirement that the sequence $\{g_n\}$ be uniformly of bounded variation and converge on all the common continuity points of g_n and on a and b, but it was left an open question whether this set of conditions would suffice for subclasses of F wider than D. It is now shown that the answer to this question is in the negative; thus, for example, for the set of the functions of F which have at most one singularity of the second kind at some fixed point u of (a, b), it is necessary to add to the above conditions the requirement of equicontinuity at u of the set $\{g_n\}$. (Received November 24, 1941.)

56. Otto Szász: *On a theorem of Hardy and Littlewood.*

The theorem under consideration asserts that if an integrable function $f(\theta)=o(\log 1/\theta)^{-1}$, then $\sum_0^n(v+1)^{-1}s_v^*=o(\log n)$. Here $s_0^*, \cdots, s_n^*$ are the values $|s_0|, |s_1|, \cdots, |s_n|$ rearranged in decreasing order, and s_n is the nth partial sum of the Fourier series for $\theta=0$. The author proves the same result under a more general assumption; applications are also given. (Received November 10, 1941.)

57. G. Szegö: *On the oscillation of differential transforms.* I.

Let $f(x)$ be a real periodic function with period 2π for which all derivatives $f^{(k)}(x)$ exist, and denote by $2N_k$ the number of the mod 2π distinct sign variations of $f^{(k)}(x)$. The following theorems hold: (1) If $N_k=O(1)$, $f(x)$ is a trigonometric polynomial. (2) If $N_k<k/\log k, f(x)$ is an integral function. (3) If $\rho>1$ and $2N_k<(k/\rho)^{1/\rho}, f(x)$ is an integral function of order $\rho/(\rho-1)$. (1) is due to G. Pólya and N. Wiener (cf. a forthcoming paper in the Transactions), (2) and (3) are refinements of certain results of these authors. (2) and (3) are best possible results since it can be shown that the conclusions of (2) and (3) are not necessarily valid if $N_k=O(1)$ or $N_k=O(k^\alpha)$, $\alpha>1/\rho$, respectively. The method used in the present paper is different from that of G. Pólya-N. Wiener. It leads also to a refinement of (2) in which from a certain limitation of N_k the analytic character of $f(x)$ in a certain strip is concluded. Finally, analogous results are obtained by replacing $f^{(k)}(x)$ by $\vartheta^k f(x)$ where $\vartheta=(1-x^2)D^2-2xD$ is Legendre's operator (cf. II of this series by E. Hille). (Received November 22, 1941.)

58. E. W. Titt: *A method for integrating the linear hyperbolic equation in three independent variables.*

This paper is concerned with Cauchy's problem with data given over a surface duly inclined to the characteristic cone. The same choice of coordinate system relative to the characteristic cone is made that the writer has used previously (Annals of Mathematics, (2), vol. 40 (1939), pp. 862–891). A solution of the homogeneous adjoint equation is obtained by integrating the elementary solution twice in a direction exterior to the cone. This quantity has a finite discontinuity in one partial derivative as in the case of Green's function for an ordinary differential equation. Integrating over a region bounded by one nap of a cone and the initial surface a formula is obtained for the unknown integrated over a two-dimensional region. Varying the vertex of the cone in order to obtain a formula for u itself seems simpler than in the case of Volterra's method. (Received November 24, 1941.)

59. W. R. Transue: *Contributions to the theory of subharmonic functions.*

First, by use of the two-constant theorem, it is shown that the maximum of a subharmonic function on a level line, $H=\lambda$, of an harmonic function, H, is a convex function of λ. A variety of facts concerning subharmonic functions are shown to be obtainable from this result. Second, a limitation on the mean of u, logarithmically subharmonic, on the circumference of circles within the unit circle is shown to induce a limitation on the mean of u^2. Finally, it is shown that, by replacing Green's function in the representation method of F. Riesz by $E_p(P, Q)\equiv\log PQ-\log OQ+R[(P/Q)+(1/2)(P/Q)^2+\cdots+(1/p)(P/Q)^p]$, subharmonic functions can be represented in the neighborhood of a point in cases where they do not possess an harmonic majorant in that neighborhood and hence where representation using Green's function is not possible. (Received October 24, 1941.)

60. W. J. Trjitzinsky: *Analytic theory of parametric linear partial differential equations.*

In two recent addresses G. D. Birkhoff brings out the significance of asymptotic developments, in the field of linear partial differential equations, for the domain of quantum mechanical ideas. He introduces formal series and makes a conjecture that actual' solutions, of an appropriate type, exist asymptotic to these series. In view of

Birkhoff's earlier work in the asymptotic theory of ordinary differential equations, as well as in consequence of certain considerations of mathematical physics, the truth of this conjecture would offhand appear as rather likely. This fact explains the purpose of the present work—taking a purely mathematical point of view, the author considers linear partial differential equations containing a parameter λ and first establishes existence of formal solutions containing those of Birkhoff as a special case. He then establishes (for second order equations) some general existence theorems, asserting existence of 'actual' solutions, which are functions asymptotic to the formal series. This theory is naturally divided into two parts—one relating to equations of elliptic type, the other referring to those of hyperbolic type. Equations of parabolic type have not been considered in the present work. (Received November 24, 1941.)

61. S. M. Ulam: *A geometrical approach to the theory of representations of topological groups.* Preliminary report.

The theorem of von Neumann on the representations of compact groups by sequences of finite matrices is proved by consideration of geometrical properties of compact convex bodies in the space of continuous (real-valued) functions on a compact space. (The use of Haar measure is avoided.) A number of related results are derived. (Received November 25, 1941.)

62. František Wolf: *On majorants of analytic functions.*

If $f(z)$ is analytic in $|x| < a, |y| < b$ and $|f(z)| \leqq M(x)$ where $\int_{-a}^{a} \log^+ \log^+ M(x) dx < \infty$, then to an arbitrary $\delta > 0$ corresponds a ϕ, dependent only on $M(x)$ and δ, but independent on the particular $f(z)$, such that $|f(z)| \leqq \phi$ for $|x| < a - \delta$, $|y| < b - \delta$. This is a generalization of a result of N. Levinson (*Gap and Density Theorems*, p. 127, Theorem XLIII). The theorem is proved by a method used by the author to prove a generalization of Phragmén-Lindelöf theorem (Journal of the London Mathematical Society, vol. 14 (1939), p. 208) which becomes a corollary of the above theorem. Another corollary is Nils Sjögren's result (Congrès des Mathématiques Scandinaves à Helsinfors, 1938): If $f(z)$ is analytic in the unit circle and such that $|f(z)| \leqq M(\arg z)$ for $1 - \epsilon < r < 1$ and $\int_0^{2\pi} \log^+ \log^+ M(\theta) \cdot d\theta < \infty$, then $|f(z)| \leqq \phi$ in $|z| \leqq 1 - \delta < 1$. ϕ does not depend on the particular $f(z)$, but only on $M(\theta)$ and δ. (Received October 25, 1941.)

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63. A. E. Engelbrecht: *Circular plates with large deflections.*

The nonlinear system of equations derived by von Kármán is used to obtain a solution for a family of thin circular plates involving radial symmetry and having a uniform moment applied at the periphery. The edge of the plate to which the external moment is applied suffers no displacement normal to the plane of the plate, but is free to move laterally. The solution is effected by expanding the deflection w and the stress function ϕ in terms of a small parameter $\epsilon = h/a$ where h is the plate thickness and a the radius of the plate. By this expansion the nonlinear system reduces to an iterative process for determining the successive terms. Satisfactory numerical results are obtained for the deflection, bending moments and direct planar stresses for plates whose maximum deflection is twice the order of the plate thickness. (Received October 21, 1941.)