PROJECTION OF THE SPACE (m) ON ITS SUBSPACE (c_0)

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In a paper in the Duke Journal, A. E. Taylor[1] remarks that it is an open question whether or not there exists a projection of the space (m), of bounded sequences, on its subspace (c_0), the space of sequences convergent to 0. In this note we make a few remarks which supplement those of Taylor on this question, and we point out that a negative answer follows from a recent result of R. S. Phillips,[2] so that the question is now settled.

Taylor shows that if a projection of the space (c), of convergent sequences, on the space (c_0) exists, it must be of norm greater than or equal to 2. This implies the same result for (m) on (c_0), since any projection of (m) on (c_0) would be in particular a projection of (c) on (c_0).

The space (c) is essentially of dimension only one greater than that of its subspace (c_0). This follows since (c) is obviously the set of all elements of the form $x=x^{(0)}+tX_1$, where $X_1=(1, 1, \cdots)$, $x^{(0)}\in(c_0)$, and t is a number. If $x=\{x_i\}$ is any element of (c), the linear functional $a(x)=t=\lim_{n\to\infty} x_n$ is of norm 1, and vanishes on the subspace (c_0). Now it is a remark of Bohnenblust[3] that for any subspace of a normed linear space L defined by the vanishing of a fixed linear functional on L, there exist projections of norm less than or equal to $2+\epsilon$, for arbitrary $\epsilon>0$. Consequently there are projections of (c) on (c_0) of norm less than or equal to $2+\epsilon$.

There are projections of (c) on (c_0) which are of norm exactly 2, as may be seen as follows. If $x=(x^{(0)}+tX_1)\in(c)$, the general form of a projection of (c) on (c_0) is

$$Px = x + t\{b_i\} = x^{(0)} + t(X_1 + \{b_i\})$$

where $\{b_i\}$ is any sequence of constants such that $\lim_{i\to\infty} b_i=-1$.[4] To calculate the norm of P, we have $\|Px\|\leqq\|x\|+|t|\cdot\sup_i |b_i|$, and $\|x\|=\|x^{(0)}+tX_1\|=\sup_i |x_i^{(0)}+t|\geqq|t|$ since $x_i^{(0)}\to 0$. Therefore $|P|\leqq 1+\sup_i |b_i|$, and because of Taylor's result this has the value

[1] *The extension of linear functionals*, Duke Mathematical Journal, vol. 5 (1939), pp. 538–547; p. 547.

[2] *On linear transformations*, Transactions of this Society, vol. 48 (1940), pp. 516–541; pp. 539–540.

[3] *Convex regions and projections in Minkowski spaces*, Annals of Mathematics, (2), vol. 39 (1938), pp. 301–308; p. 308.

[4] See Taylor, op. cit.

2 for all projections such that $|b_i| \leqq 1$ for all i. In particular the projection $Px = x - a(x) \cdot X_1 = x^{(0)}$, obtained by taking $b_i = -1$ for all i, is of norm 2.

Corresponding to any sequence $\{e_i\}$, where, for each i, e_i is either $+1$ or -1, let us define a space $c_{\pm}\ldots$ as the set of all elements of the form $x + t\{e_i\}$, where $x \in (c_0)$. Any space $c_{\pm}\ldots$ is equivalent[5] to (c) under the automorphism $\{x_i\} \leftrightarrow \{e_i x_i\}$ of (m). Also (c_0) is a subspace of $c_{\pm}\ldots$, since (c_0) is invariant under the automorphism. There are obviously projections of norm 2 of any $c_{\pm}\ldots$ on (c_0), similar to the projections of (c) on (c_0).

If L and L' are any two linear subspaces of (m), we define the *sum* $L + L'$ as the linear subspace of (m) which consists of all elements of the form $x + x'$, where $x \in L$, $x' \in L'$. As usual we denote the closure in (m) of any subspace by a horizontal line over the symbol representing the subspace.

For any set of n linearly independent sequences $X_j = \{e_{ij}\}$, $e_{ij} = +1$ or -1, $j = 1, 2, \cdots, n$, let l_n denote the n dimensional space of all linear combinations $\sum_j t_j X_j$. Suppose the X_j's are such that l_n and (c_0) do not intersect, except in the origin. Then we may define a subspace (c_n) of (m) as the space $(c_0) + l_n$. Similarly, for any countable sequence of X_j's, any finite number of which are linearly independent, we denote the space of all finite linear combinations $\sum_j t_j X_j$ by l_f, and we define $l_\infty = \bar{l}_f$ $(\infty = \aleph_0)$. If l_f and (c_0) intersect only in the origin, we define

$$(c_\infty) = \overline{(c_0) + l_\infty} = \overline{(c_0) + l_f}\ .$$

(If l_∞ and (c_0) intersect only in the origin, each $x \in (c_0) + l_\infty$ has a unique decomposition $x = x^{(0)} + x^{(\infty)}$, $x^{(0)} \in (c_0)$, $x^{(\infty)} \in l_\infty$; otherwise not. The space $(c_0) + l_\infty$ is not necessarily closed.)

Let $X_1 = \{e_{i1}\}$, $X_2 = \{e_{i2}\}$ $\cdots$ be the rows of an infinite matrix $\{\{e_{ij}\}\}$; let $\{\{e_{ij}\}\}_n$ denote the matrix of the first n rows of $\{\{e_{ij}\}\}$. Then any space (c_n) or (c_∞) is determined by a matrix $\{\{e_{ij}\}\}_n$ or $\{\{e_{ij}\}\}$.

THEOREM 1. *In any space (c_n) [or (c_∞)] such that for every column C_k which appears in the matrix $\{\{e_{ij}\}\}_n$ [for each n] one of $+C_i$, $-C_i$ is identical with C_k for an infinite number of values of i, as $i \to \infty$, there exists a projection of (c_n) or (c_∞) on (c_0), which is of norm* 2.

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[5] Two normed linear spaces are isomorphic if there exists a 1-1 transformation T between them which is linear (that is, distributive and continuous) in both directions; they are equivalent if in addition $|T| = |T^{-1}| = 1$.

PROOF. If $x = x^{(0)} + \sum_{j=1}^{j=n} t_j X_j$, $x^{(0)} \in (c_0)$, define $Px = x^{(0)}$. Then

$$\|Px\| = \left\| x - \sum_j t_j X_j \right\| \leqq \|x\| + \left\| \sum_j t_j X_j \right\| = \|x\| + \sup_i \left| \sum_j t_j e_{ij} \right|.$$

By the hypothesis on the matrix $\{\{e_{ij}\}\}_n$, and since if $x^{(0)} = \{x_i^{(0)}\}$, $x_i^{(0)} \to 0$, $\|x\| = \sup_i |x_i^{(0)} + \sum_j t_j e_{ij}| \geqq \max_i |\sum_j t_j e_{ij}|$. This implies $\|Px\| \leqq 2\|x\|$ for any x of (c_n) or of the dense linear set in (c_∞). In the latter case P has a unique continuous extension to (c_∞), with range (c_0), and the norm is preserved. This verifies the theorem.

THEOREM 2. *For every space* (c_n), $1 \leqq n \leqq \infty$, *there is a projection of* (c_n) *on* (c_0) *which is of norm* 2.

PROOF. Let (c_n) be determined by $X_1 = \{e_{i1}\}$, $X_2 = \{e_{i2}\}$, $\cdots$, as above. Consider first the case of any finite n. Since only a finite number of different n-element columns are possible, there can be only a finite number of columns C_k in the matrix $\{\{e_{ij}\}\}_n$ which are not infinitely repeated. Since X_1 and X_2 are linearly independent, and l_2 intersects (c_0) only in the origin, the matrix $\{\{e_{ij}\}\}_2$ evidently satisfies the hypothesis of Theorem 1. Consider $(c_3) = (c_0) + l_3$. By altering at most a finite number of coordinates of X_3, the matrix $\{\{e_{ij}\}\}_3$ can be made to satisfy the hypothesis. Let the altered sequence X_3 be X_3'; then $X_3' = X_3 + y^{(0)}$, where $y^{(0)} \in (c_0)$. Therefore the space $(c_3)'$ determined by X_1, X_2, X_3' coincides with (c_3). Proceeding in similar fashion, we see that any space (c_n) or $(c_0) + l_f$ may be represented by a matrix $\{\{e_{ij}\}\}_n$ or $\{\{e_{ij}\}\}$ which *does* satisfy the hypothesis of Theorem 1. Our theorem is thus proved.

THEOREM 3. *There exists a matrix* $E = \{\{e_{ij}\}\}$, *such that*: (1) *the corresponding subspace* l_∞ *of* (m) *intersects* (c_0) *only in the origin, and is equivalent to the space* $l_{1,\infty}$ *of absolutely convergent series*; (2) $(c_0) + l_\infty$ *is closed in* (m), *and* l_∞ *and* (c_0) *are complementary subspaces in* $(c_\infty) = (c_0) + l_\infty$; (3) *the projection of* (c_∞) *on* (c_0) *according to the complementary subspace* l_∞ *of* (c_0) *in* (c_∞) *is of norm* 2.

PROOF. Let $+$ and $-$ represent $+1$ and -1. The matrix E is

$$\begin{matrix} + & - & + & - & + & - & + & - & \cdots \\ + & + & - & - & + & + & - & - & \cdots \\ + & + & + & + & - & - & - & - & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

(The kth row of E consists of alternately 2^{k-1} plus signs and 2^{k-1} minus signs, for all k.) In this matrix, obviously *all* possible finite combinations of $+$'s and $-$'s occur in the columns. Thus we have for any n $\|\sum_1^n t_j X_j\| = \sum_1^n |t_j|$. The space l_∞ determined by E is therefore equivalent to $l_{1,\infty}$. Suppose that (c_0) and (l_∞) have a common element x. Then there is a sequence $\{x_n\}$, $x_n = \sum_1^n c_{j,n} X_j$, which converges to x. Since $x \in (c_0)$, by the construction of E we must have $\sum_i^n |c_{j,n}| \to 0$ as $n \to \infty$. But this implies $x_n \to 0$, $x = 0$. This verifies statement (1) of Theorem 3; statement (3) follows by Theorem 1. Statement (2) then follows by a lemma of F. J. Murray, that (for Banach spaces) the existence of projections and of complementary manifolds are equivalent.[6]

THEOREM 4. *There exists a subspace of* (m) *equivalent to the space* (C) *of continuous functions, such that*: (1) *the subspace* (C) *of* (m) *and* (c_0) *intersect only in the origin*; (2) $(C)+(c_0)$ *is closed in* (m); (3) *the projection of* $(C)+(c_0)$ *on* (c_0) *according to the complementary subspace* (C) *of* (c_0) *in* $(C)+(c_0)$ *is of norm* 2.

PROOF. Let $\{r_i\}$ be the set of all rational numbers of the interval $(0, 1)$, in their usual enumeration. Then any $x = x(t)$ of (C) is uniquely determined by the set of values $\{x(r_i)\}$, while (m) may be regarded as the set of all bounded real-valued functions $x = \{x_i\} = \{x(r_i)\}$ on the set $\{r_i\}$. Clearly this correspondence of each $x(t) \in (C)$ to its set of values $\{x(r_i)\}$ in (m) is an equivalence. Furthermore it is obvious that if $\lim_{i\to\infty} x(r_i) = 0$, $x(t) \equiv 0$. Therefore (C) according to this imbedding intersects (c_0) only in 0.

If $x = \{x(r_i)\}$ corresponds to $x(t) \in (C)$, and $x^{(0)} = \{x^{(0)}(r_i)\}$ belongs to the subspace (c_0) of (m), then $\sup_i |x(r_i) + x^{(0)}(r_i)| \geqq \max |x(t)| = \sup_i |x(r_i)|$ since $x^{(0)}(r_i) \to 0$, and $x(t)$ is continuous. Thus $\|x + x^{(0)}\| \geqq \|x\|$, and the operation defined by $Q(x + x^{(0)}) = x$ is a projection of norm 1 of $(C)+(c_0)$ on (C). The operation $P = I - Q$ is a projection of $(C)+(c_0)$ on (c_0), and $|P| = |I - Q| \leqq |I| + |Q| = 2$. The proof of statement (2) of Theorem 4 is the same as the first part of the proof of Theorem 6, below.

It is a well known theorm of Banach that every separable Banach space is equivalent to a subspace of (C).[7] Therefore by Theorem 4, if (B) is any separable Banach space, there is a subspace of (m) equivalent to (B), such that the projection of $(c_0)+(B)$ on (c_0) through (B) is of norm 2.

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[6] See F. J. Murray, Transactions of this Society, vol. 41 (1937), pp. 138–139.

In a normed linear space L, two linear subspaces L_1 and L_2 are called *complementary* manifolds or subspaces if (1) they are closed in L, and intersect only in 0; (2) every $x \in L$ has a decomposition $x = x_1 + x_2$, where $x_1 \in L_1$, $x_2 \in L_2$. It follows by assumption that this decomposition is unique.

[7] S. Banach, *Opérations Linéaires*, p. 185, Theorem 9.

The result of Theorem 2 may be generalized, by removing the restriction that the e_{ij}'s may have only the values $+1$ or -1. Let $X_1=\{e_{i1}\}$ be any element of $(m)-(c_0)$. Let $S_1=[tX_1]$ be the space of all elements tX_1, and let $B_1=(c_0)+S_1$. If P is a projection of B_1 on (c_0), define a sequence of functionals $\{y_i(x)\}$ by $Px=x^{(0)}+tX_1+\{y_i(x)\}$, where $x=x^{(0)}+tX_1$, $x^{(0)}\in(c_0)$. The functionals $y_i(x)$ are obviously linear, and $y_i(x^{(0)})=0$ for all i and $x^{(0)}\in(c_0)$. Let $y_i(X_1)=a_{i1}$. Then $tPX_1=tX_1+t\{a_{i1}\}\in(c_0)$, and we see that the general form of a projection of B_1 on (c_0) is $Px=x^{(0)}+t\{a_{i1}+e_{i1}\}$, where $\{a_{i1}\}$ is any sequence of constants such that $\lim_{i\to\infty}(a_{i1}+e_{i1})=0$.

If $X_1=\{e_{i1}\}$ as above, let $\limsup_i |e_{i1}|=a$. Define $X_1'=\{e_{i1}'\}$ by $e_{i1}'=e_{i1}$ if $|e_{i1}|\leqq a$, $e_{i1}'=a$ sign e_{i1} if $|e_{i1}|\geqq a$. Then $(X_1-X_1')=\{e_{i1}-e_{i1}'\}\in(c_0)$, and the space $B_1'=(c_0)+S_1'$ is the same as B_1. Thus without loss of generality we may always assume $|e_{k1}|\leqq a=\limsup_i|e_{i1}|$ for all k. This assumption is made in the following paragraph.

There are projections of any space B_1 on (c_0) which are of norm 2. For if $Px=x+t\{a_{i1}\}$, then $\|Px\|\leqq\|x\|+|t|\cdot\sup_i|a_{i1}|$. Since $\|x\|=\|x^{(0)}+t\{e_{i1}\}\|=\sup_i|x_i^{(0)}+te_{i1}|\geqq\sup_i|te_{i1}|$, we see that $|P|$ is 2 for all P such that $|a_{i1}|\leqq|e_{i1}|$ for all i. (Note that in no case can $|P|$ be *less* than 2.)

For any finite or countable set of linearly independent sequences $X_j=\{e_{ij}\}$, (e_{ij}'s not restricted to ±1), let S_n or S_f respectively denote the subspace of all linear (finite) combinations $\sum_j t_jX_j$. As before, suppose that S_n or S_f does not intersect (c_0), except in the origin. Define $B_n=(c_0)+S_n$, $B_f=(c_0)+S_f$, $B_\infty=\overline{B}_f$. The general form of a projection of B_n on (c_0) is $Px=x+\{\sum_j t_ja_{ij}\}$ where $x=x^{(0)}+\sum_j t_jX_j$, $x^{(0)}\in(c_0)$, and the a_{ij}'s are any constants such that $\lim_{i\to\infty}(a_{ij}+e_{ij})=0$ for each j. This follows by additivity of P, since P must be in particular a projection of each of the subspaces $(c_0)+[tX_j]$ of B_n on (c_0).

THEOREM 5. *If W is any separable Banach subspace of (m) such that $W\supset(c_0)$, there is a projection of W on (c_0) which is of norm* 2.

PROOF. It follows easily from the hypothesis that W is separable, that there exists a sequence $\{X_j\}$ such that either W is a space B_n (n finite), or

$$W=B_\infty=\overline{(c_0)+S_f}.$$

(Conversely, any space B_n or B_∞ is separable.)

Consider any space B_n or B_f determined by a matrix $E=\{\{e_{ij}\}\}$ which is such that the e_{ij}'s have only a finite number N of different values. By an argument similar to the proofs of Theorems 1 and 2, we see that there is a projection of B_n or B_f on (c_0) which is of norm 2. This projection is specifically the one given by the general form above when $a_{ij}=-e_{ij}$ for all i, j (after the e_{ij}'s have been suitably altered).

For any space B_n or B_f, without loss of generality we may assume that $\|X_j\|=1$ for all j. For each integer N, divide the interval $0\leqq u\leqq 1$ into N subintervals $0\leqq u\leqq N^{-1}$, $N^{-1}<u\leqq 2N^{-1}, \cdots, (k-1)N^{-1}<u \leqq kN^{-1}, \cdots, (N-1)N^{-1}<u\leqq 1$. If $|e_{ij}|$ lies in the kth of these intervals, let $k_{ij}=kN^{-1}$ sign e_{ij}. In this way we associate with the matrix $\{\{e_{ij}\}\}_n$ of B_n a matrix $\{\{k_{ij}\}\}_n$, in which the k_{ij}'s have only the values $\pm kN^{-1}$, $k=1, \cdots, N$. By a finite number of alterations, the matrix $\{\{k_{ij}\}\}_n$ may be changed into a matrix $\{\{k_{ij}^{(N)}\}\}_n$ in which every column is infinitely repeated; this requires addition of integral multiples of $\pm N^{-1}$ to a finite number of the k_{ij}'s. Alter the matrix $\{\{e_{ij}\}\}_n$ by adding the same multiplies of $\pm N^{-1}$ to the corresponding e_{ij}'s; we thus obtain a new matrix $\{\{e_{ij}^{(N)}\}\}_n$ which represents the same space B_n, and which has $\{\{k_{ij}^{(N)}\}\}_n$ for its associated matrix. Let the rows of $\{\{e_{ij}^{(N)}\}\}$ be denoted by $\{X_j^{(N)}\}$; those of $\{\{k_{ij}^{(N)}\}\}$ by $\{Y_j^{(N)}\}$.

If $x\in B_n$, for each N let $x=x^{(0,N)}+\sum_{j=1}^{n} t_jX_j^{(N)}$, where $x^{(0,N)}\in(c_0)$. Let $x^{(0,N)}=P^{(N)}x$. Let N take on only the succession of values 2^ν, $\nu=1, 2, 3, \cdots$. Then evidently we may make the required alterations in the matrices $\{\{k_{ij}^{(N)}\}\}_n$ (and $\{\{e_{ij}^{(N)}\}\}_n$) in such a way that $\lim_{N,N'}\|X_j^{(N)}-X_j^{(N')}\|=0$ for each j; that is, $\{X_j^{(N)}\}$ is a Cauchy sequence for each j. If $x=x^{(0,N)}+\sum_j t_jX_j^{(N)}$ and $x=x^{(0,N')}+\sum_j t_jX_j^{(N')}$,[8] then $0=x^{(0,N)}-x^{(0,N')}+\sum_j t_j(X_j^{(N)}-X_j^{(N')})$, and

$$\|x^{(0,N)}-x^{(0,N')}\|=\|\sum_j t_j(X_j^{(N)}-X_j^{(N')})\|.$$

Therefore for each $x\in B_n$, $\{x^{(0,N)}\}$ is a Cauchy sequence of elements of (c_0).

Let $x^{(0)}=\lim_N x^{(0,N)}$; and define $Px=x^{(0)}$. Then

$$\|P^{(N)}x\| = \left\|x - \sum_j t_jX_j^{(N)}\right\| = \left\|x - \sum_j t_jY_j^{(N)} + \sum_j t_j(Y_j^{(N)} - X_j^{(N)})\right\|$$

$$\leqq \|x\| + \left\|\sum_j t_jY_j^{(N)}\right\| + \left\|\sum_j t_j(Y_j^{(N)} - X_j^{(N)})\right\|.$$

We also have

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[8] The t_j's do not change with N, because in the expression of any x, for each succeeding N the finite alterations can be compensated by changing $x^{(0,N)}$—and for each system $\{X_j^{(N)}\}$, the expression of x is unique (by the hypothesis that S_n intersects (c_0) only in the origin).

$$\|x\| = \left\| x^{(0,N)} + \sum_j t_j X_j^{(N)} \right\|$$

$$\geqq \left\| x^{(0,N)} + \sum_j t_j Y_j^{(N)} \right\| - \left\| \sum_j t_j (Y_j^{(N)} - X_j^{(N)}) \right\|$$

$$\geqq \left\| \sum_j t_j Y_j^{(N)} \right\| - \left\| \sum_j t_j (Y_j^{(N)} - X_j^{(N)}) \right\|$$

by the triangle property of the norm, and since each column of $\{\{k_{ij}^{(N)}\}\}_n$ is infinitely repeated. Therefore $\|P^{(N)}x\| = \|x^{(0,N)}\| \leqq 2\|x\| + 2\|\sum_j t_j(Y_j^{(N)} - X_j^{(N)})\|$. Taking the limit as $N \to \infty$ $(\nu \to \infty)$ in this inequality, we see that $\|Px\| \leqq 2\|x\|$ (since $\|P^{(N)}x - Px\| \geqq |\|P^{(N)}x\| - \|Px\|| \to 0$, $\|P^{(N)}x\| \to \|Px\|$).

The operation P, defined for each $x \in B_n$ or B_f, is clearly a projection of norm 2 of B_n or B_f on (c_0), and our theorem is proved. (In case of B_f, as in Theorem 1 P has a unique extension to $\overline{B}_f = B_\infty = W$.)

The foregoing discussion has produced instances of the existence of projections defined on subspaces of (m) to (c_0), culminating in Theorem 5. Using the result of Phillips mentioned above, we now show that there is no projection defined on (m) to (c_0).[9] This result is perhaps rather surprising, since in all of our previous instances, not only did projections exist, but also there were projections of norm exactly 2. (One might expect to be able to find easily a succession of larger and larger subspaces containing (c_0), for which the greatest lower bounds of the norms of projections would increase unboundedly.) Theorem 5 shows that the nonexistence of a projection on (m) to (c_0) is due in an essential way to the *inseparability* of (m).

Phillips' result is that there is no projection on (m) to (c). It follows easily from this that there is no projection on (m) to (c_0), in view of the following simple theorem.

THEOREM 6. *Suppose that l_1, l_2 are any pair of complementary subspaces in a Banach space L, and that l_2', l_2'' are any pair of complementary subspaces in l_2. Let $L_1 = (l_1 + l_2')$. Then L_1 is closed in L, and L_1 and $L_2 = l_2''$ are complementary subspaces in L.*

PROOF. Let P be the projection of L on l_1 according to l_2. Suppose that $\{x_n\}$ is any sequence of L_1, such that $x_n \to x$ in L, and let $x_n = x_{n1} + x_{n2}'$, where $x_{n1} \in l_1$, $x_{n2}' \in l_2'$. Then $Px_n = x_{n1} \to Px = x_1 \in l_1$, and $(I-P)x_n = x_{n1} + x_{n2}' - x_{n1} \to (I-P)x = x_2' \in l_2'$, by the continuity of P and of $I-P$. Therefore $x_n = x_{n1} + x_{n2}' \to x_1 + x_2' = x$, so that $x \in L_1$, and L_1 is closed. Moreover, l_1 and l_2' are complementary subspaces in L_1.

[9] "Projection *of* (m) *on* (c_0)" of course means the same as "projection *on* (m) *to* (c_0)."

By hypothesis, for any $x \in L$, $x = x_1 + x_2 = x_1 + x_2' + x_2''$, where $x_1 \in l_1$, $x_2' \in l_2'$, $x_2'' \in l_2''$, and this decomposition is unique. Therefore the decomposition $x = (x_1 + x_2') + x_2''$ according to L_1 and L_2 also is unique. This verifies Theorem 6.

If there were a projection of (m) on (c_0), there would be a complementary subspace l_2 to $l_1 = (c_0)$ in (m). The element $X_1 = (1, 1, \cdots)$ $\in (m)$ would have a decomposition $X_1 = x^{(0)} + X$, $x^{(0)} \in (c_0)$, $X \in l_2$. Consider the one-dimensional subspace consisting of all elements of the form tX. By the Hahn-Banach theorem, there exists a projection of norm 1 on any one-dimensional subspace. Thus if $l_2' = \{tX\}$, there exists a complementary subspace l_2'' in l_2. The space $L_1 = (l_1 + l_2')$ of Theorem 6 is (c), and by Theorem 6 and the lemma of Murray already mentioned, there would exist a projection of (m) on (c), contradicting the result of Phillips. Thus there can be no projection of (m) on (c_0).

By an argument identical with that used by Phillips to prove the nonexistence of a projection of (m) on (c), it may also be shown directly that there is no projection of (m) on (c_0). Theorem 6 will be required, however, for another remark to be made below. (It should perhaps be mentioned that (c) is isomorphic to (c_0).[10])

By Theorem 5 and since there is no projection of (m) on (c_0), there is no projection of (m) on any separable subspace $W \supset (c_0)$. (For if Q were a projection of (m) on W, P a projection of W on (c_0), then PQ would be a projection of (m) on (c_0).) The following question now arises: Does there exist a closed linear subspace L of (m), $L \supset (c_0)$, such that there is no projection of (m) on L, and no projection of L on (c_0)? Of course by Theorem 5, such a space L cannot be separable.

An extension theorem found in the paper by Phillips which has been cited[11] is as follows. Let X be any linear subspace of a Banach space Z, and let U be any linear transformation on X to (m). If $y = \{y_i\} = Ux$, the functionals $y_i(x)$ are obviously linear, and by the Hahn-Banach theorem they may be extended to Z with preservation of the norm. Let $\bar{z}_i(z)$ denote the extension of $y_i(x)$. Then $U_1 z = \{\bar{z}_i(z)\}$ is a linear transformation defined on Z with range (m), which coincides with U on X, and $|U_1| = |U|$. As a consequence of this, if X is isomorphic to any subspace (B) of (m), and if there exists any Banach space $W \supset X$ such that there is no projection of W on X, then there is no projection of (m) on (B). For if U is the isomorphism, U_1 its extension on W, and if Q were the projection of (m) on (B), then $U^{-1}QU_1$ would be a projection of W on X.

[10] Banach, op. cit., p. 181.

[11] P. 538, Theorem 7.1, Corollary 7.2.

It follows immediately by the previous paragraph that not only is there no projection of (m) on (c_0) in the usual orientation of (c_0) with respect to (m), but also that there is no projection of (m) on (c_0) in any imbedding whatever of (c_0) in (m). (Let W be (m), let X be (c_0) in the usual imbedding, let (B) be (c_0) in an arbitrary imbedding in (m), and take U to be the identity on (c_0) to (c_0).) Also, since as has been shown by Phillips[12] there exists no projection of (C_1) (the space of functions having only discontinuities of the first kind) on (C), it follows that there is no projection of (m) on (C) in the imbedding of Theorem 4, or in any other imbedding.

Another interesting consequence of Phillips' extension theorem is that for every Banach space $Z \supset (m)$, there is a projection of norm 1 on Z to (m). (Extend to Z the identity transformation of $(m) \subset Z$ into (m).) Also, consider the finite dimensional spaces $l_{\infty,n}$ of sequences $x=(x_1, \cdots, x_n)$, with norm $\|x\| = \max_i |x_i|$. In a similar way, it follows that there is a projection of norm 1 on any Banach space $Z \supset l_{\infty,n}$ to $l_{\infty,n}$. There is probably some connection between this and the result of Theorem 5, since $(c_0) \supset l_{\infty,n}$ for every n. A question which arises here is whether the most general separable Banach space having (c_0) as a subspace is contained in (m) in the same relationship to (c_0) as in Theorem 5. Anyway, if B is any separable Banach space having a subspace equivalent to (c_0), there is a projection of norm 2 of B on (c_0); for let U be the identity on (c_0) in B to (c_0) in (m), extend U to be U_1 on B to (m) by Phillips' theorem, and let the proper range of U_1 be W'. By continuity of U_1, W' is separable. Let P' be a projection of norm 2 on W' to (c_0), as given by Theorem 5. Then $U^{-1}P'U_1$ is the required projection of norm 2 on B to (c_0).

Any separable Banach space Y is equivalent to a subspace (B) of (m).[13] The question of whether there is an infinite dimensional, separable Banach subspace (B) in (m), such that there *is* a projection on (m) to (B), is equivalent to the question of whether there is a separable Banach space Y, such that for every Banach space $W \supset Y$, there is a projection on W to Y. A further question is whether such a subspace is necessarily reflexive. ((c_0) is not reflexive.)

Let Y be any separable Banach space; imbed Y in (C) and (C) in (m). If there is no projection on (C) to Y, then there is no projection on (m) to Y in any imbedding. If there is a projection on (C) to Y, then using Theorem 6 and the facts that any two complementary subspaces of the same closed linear subspace are isomorphic, and that

[12] Loc. cit., p. 539.

[13] A direct imbedding is given by Phillips, loc. cit., p. 524.

there is no projection on (m) to (C), it may be shown that at least either there is no projection on (m) to Y, or else there is no projection on (m) to the complementary subspace of Y in (C). (An illustration of the case where there is no projection on (m) to the complementary subspace in (C) is provided by the case of a finite dimensional $Y \subset (C)$.)

In a paper in preparation on the extension of linear transformations, the writer intends to discuss the questions indicated above, and related questions.

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SEQUENCES OF STIELTJES INTEGRALS[1]

H. M. SCHWARTZ

Statement of results. Sequences of Riemann-Stieltjes integrals[2] have as yet been little studied, only the following fundamental results being known.

THEOREM A (Helly [2]). *Let $g_n(x)$ $(n=1, 2, \cdots)$ be an infinite sequence of real functions defined in the finite closed interval $I=(a, b)$ which satisfy the following two conditions:*

$$\text{(1)} \qquad \text{Total variation of } g_n \text{ in } I \equiv V_I(g_n) \leqq M, \qquad M \text{ a fixed constant,}$$

$$\text{(2)} \qquad g_n \to g \text{ on } I, \qquad n \to \infty;$$

then for any function $f(x)$ continuous in I, we have[3]

$$\text{(3)} \qquad \int f dg_n \to \int f dg.$$

THEOREM B (Shohat [3]). *Let $\{g_n\}$ be a sequence of functions monotonic and uniformly bounded in I and such that*

$$\text{(4)} \quad g_n \to g \text{ on } E, \quad E \text{ a set dense on } I \text{ and including the end points } a, b \text{ of } I,$$

where g is a monotonic function (all the functions g_n, g monotonic in the same sense); then we have (3) *for any function $f(x)$ for which*

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[1] Presented to the Society, January 1, 1941.

[2] A discussion of such integrals with references is to be found in [1]. (Numbers in brackets refer to the bibliography.)

[3] When the limits of integration are omitted, it is to be understood that they are the end points a, b of I.