well-chosen list of suggested texts for further reading. At the end of the book there are tables of Greek and German letters, and a list of some of the more important symbols used in the text.

On the whole the book is well written and the typography excellent. The reviewer noticed a few misprints and even an occasional slip. However, these are mostly of a minor nature and should cause little or no confusion to the student. The book is a valuable and timely addition to the available texts on algebra.

NEAL H. MCCOY

Gap and Density Theorems. By Norman Levinson. (American Mathematical Society Colloquium Publications, vol. 26.) New York, American Mathematical Society, 1940. 8+246 pp. $4.00.

In this book the author confines himself to a detailed study of a few salient topics in gap and density theory; he does not attempt to write a systematic treatise on the subject. The book is in form essentially a collection of research papers; it achieves unity principally through the author's repeated application of similar methods to a variety of problems. Most of the contributions to gap and density theory contained in the book are the author's own work, some of the most remarkable of them being published here for the first time. The principal topics treated are, on the one hand, the influence of the distribution of a sequence of numbers $\{\lambda_n\}$ on the closure properties of the sequence $\{e^{i\lambda_n x}\}$, and the closely related topic of the influence of the distribution of the λ_n on the growth of analytic functions which vanish or are otherwise restricted at points $z=\lambda_n$; and, on the other hand, general Tauberian theorems involving gap conditions. Among the topics not treated are, for example, the Paley-Wiener theory of "pseudoperiodic" functions, and Bochner's generalizations of it. The extensive "classical" theory connecting gap or density properties of a sequence $\{\lambda_n\}$ with the position of the singularities of the function having the Dirichlet series $\sum a_n e^{-\lambda_n x}$ is represented by one theorem. The author expects his readers to be familiar with approximately the amount of information contained in Titchmarsh's *Theory of Functions*. Familiarity with the Colloquium Publication of Paley and Wiener is not prerequisite, but would be advantageous for a reader. The author collects in an appendix the auxiliary theorems which he most frequently uses. His principal tools are such things as Jensen's theorem, Carleman's theorem which is its analogue for a half-plane, Phragmén-Lindelöf theorems, and the L^2 theory of Fourier transforms; these he combines in ingenious and often unexpected ways.

Many of the proofs are long and involve elaborate calculations. However, the author has been helpful in many places by prefacing his longer proofs with applications of his methods to simpler problems, so that the reader can see the essential ideas unobscured by details. At times, on the other hand, he demands considerable background from the reader; for example, the "obvious" statement at the beginning of Chapter V seems to be obvious principally because it is a known true theorem.

On many of the problems which he discusses, the author says the last word, since the problems are solved by theorems which are shown by examples to be "best possible." In most cases he generalizes existing theorems by constructing entirely new proofs, rather than by generalizing existing proofs; his methods have allowed him to attain wide generality and great precision. Anyone who is interested in Fourier integrals, the growth of analytic functions, or Tauberian theorems, should profit from studying the author's methods of attack. There is no reason for supposing that their power is yet exhausted.

A chapter-by-chapter résumé follows. In Chapters I and II the central problem is the closure of a sequence $\{e^{i\lambda_n x}\}$. Such a sequence is said to be closed[1] L^p if $\int_{-\pi}^{\pi} f(x)e^{i\lambda_n x}\,dx=0$ for all n, and $f(x)\in L^p$, imply $f(x)=0$ almost everywhere. As Szász has pointed out, studying the closure of a sequence $\{e^{i\lambda_n x}\}$ amounts to studying the zeros of entire functions of the form $\int_{-\pi}^{\pi} f(t)e^{ixt}dt$; and as Paley and Wiener observed, closure theorems have as corollaries theorems stating that a function with gaps in its sequence of Fourier coefficients cannot vanish over too long an interval without vanishing identically. The first two chapters contain a variety of closure and gap theorems, some in sharper form than when they were previously published. The few that are not due to the author in their present degree of generality are due to Miss Cartwright. Chapter III contains the theorems connecting the distribution of the zeros of an entire function of exponential type with its rate of growth along the real axis, used in proving the closure theorems of Chapters I and II. The essential idea is that if the function does not grow too fast along the real axis most of its zeros must lie near the real axis, and the sequence of zeros has a density in the right and left half-planes; the author's theorems are more precise than earlier results of this character.

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[1] The author follows Paley and Wiener, who interchange the customary terms. Most authors would write "complete in $L^{p'}$," where $p'=p/(p-1)$, instead of "closed L^p." Since the properties of closure in the usual sense and in the author's are equivalent, no great harm is done.

Chapter IV is devoted to non-harmonic Fourier series. If a sequence $\{e^{i\lambda_n x}\}$ is closed L^p, every function of L^p can be represented as the limit (in the topology of L^p) of a sequence of linear combinations of the functions $e^{i\lambda_n x}$. It was observed by Paley and Wiener that the restriction $|\lambda_n - n| \leqq c$, with sufficiently small c, will ensure in additon that the sequence $\{e^{i\lambda_n x}\}$ will serve to expand functions into series preserving many of the properties of ordinary Fourier series; such series they called non-harmonic Fourier series. Paley and Wiener showed that there is an L^2 theory if $c < 1/\pi^2$. Levinson showed, by different methods, that there is an L^p theory $(1 < p \leqq 2)$ if $c < (p-1)/(2p)$, but not necessarily if $c \leqq (p-1)/(2p)$. His results are reproduced in Chapter IV, together with an interesting inequality of Hardy and Levinson which is connected with the problem.

Chapter V contains various developments of the author's results which state (roughly) that if a function is "small at infinity" and its Fourier transform coincides with an analytic function over an interval, then its Fourier transform continues to coincide with the analytic function over the largest interval of the real axis on which that function continues to be analytic. Such results generalize gap theorems stating that if the Fourier transform vanishes over an interval it vanishes identically (the analytic function being in this case identically zero). Chapter VI contains very precise estimates for the rate of growth of an entire function of exponential type whose zeros have a density; these are applied to prove the classical theorem of Pólya connecting the density of the exponents of a Dirichlet series with the position of its singular points on its abscissa of convergence. In Chapter VII the estimates are combined with results from Chapter V to establish conditions under which the rate of growth of an analytic function on a sequence of points on or near a line determines its rate of growth along the whole line. Theorems of this character were obtained by V. Bernstein from the theory of Dirichlet series; the author proves Bernstein's theorems by simpler and more natural methods, and then develops a method of proof in which the results of Chapter V can be used to make the results much more precise.

The greater part of the material of Chapters I–VII has been published before; Chapters VIII and IX are new material. Pólya set as a problem the theorem that an entire function of order one and zero type is constant if it is bounded at the integers. The author replaces the integers by a sequence $\{\lambda_n\}$ sufficiently near the integers; in particular, if

$$\lambda_n - n = O\{n/(\log n)^{1+\delta}\}$$

as $n\to\infty$, for some positive δ, the theorem remains true; the theorem of Chapter VII is still more general, but too complicated to reproduce here. Remarkably enough, the condition above is approximately "best possible," since the author shows that there are sequences $\{\lambda_n\}$ satisfying the condition $\lambda_n - n = O(n/\log n)$, but such that there exist entire functions of zero exponential type bounded on the sequence and not constant; thus the existence of a positive density for the sequence $\{\lambda_n\}$ is not enough to make the theorem true. The proof of the theorem is based on a powerful inequality for analytic functions. Let $M(x)$ be a positive even function, decreasing for increasing $|x|$, and let $M(x)\to\infty$ as $x\to 0$. Let $f(z)$ be analytic in the rectangle $|x|\leqq a$, $|y|\leqq b$, and let $|f(x+iy)|\leqq M(x)$ for $x+iy$ in the rectangle. If $\int_0^1 \log\log M(x)\,dx$ converges, then $|f(x+iy)|\leqq C$ for $|x|\leqq a$ and $|y|\leqq b(1-\delta)$, where C depends only on δ and on $M(x)$. This theorem seems likely to prove useful in many problems other than that to which it is applied in this chapter. Chapter VIII is devoted to showing that the results of Chapter VII (both the theorem just quoted and the results on functions of zero type) are best possible. This is done by constructing a suitable function of zero type by means of its representation as an infinite product; this is by no means a simple task (it requires 33 pages).

Chapters X, XI, and XII are devoted to gap Tauberian theorems; Chapter X is substantially the same as a recent paper of Levinson's, while the other two chapters consist of new material. A Tauberian theorem for Dirichlet series states that if $f(x)=\sum_1^\infty a_k e^{-\mu_k x}$ for $x>0$ and $f(x)\to s$ as $x\to 0+$, then under suitable conditions $\sum a_k = s$. Most such theorems involve restrictions on the size of the a_k; but a classical theorem of Hardy and Littlewood states that if $\mu_{k+1}/\mu_k \geqq \theta > 1$, no restriction on the a_k is necessary. In the language of Wiener's general Tauberian theorems, one has to consider when, from

$$f(x) = \sum_1^\infty a_n \int_{\lambda_n - x}^\infty K(y)dy, \qquad \int_{-\infty}^\infty K(y)dy = 1,$$

and $f(x)\to s$ as $x\to\infty$, one can deduce that $\sum a_n = s$. [The Dirichlet series case corresponds to $K(y)=e^y \exp(-e^y)$.] In Wiener's theory the a_n are restricted in size and $K(y)$ is required to have a Fourier transform which does not vanish anywhere. For a gap Tauberian theorem, one might expect to be able to omit restrictions on the a_n provided that $\lambda_{n+1}-\lambda_n \geqq L > 0$. The author shows by an example that there are kernels $K(y)$ [e.g., $e^{-y^2/2}$] whose Fourier transforms do not vanish but for which such a Tauberian theorem does not hold. In Chapter X he shows that there is a gap Tauberian theorem for $K(y)$ provided that its Fourier transform satisfies sufficiently strong additional conditions. His proof, like Wiener's proof of the Hardy-Littlewood theorem, depends on reducing the Tauberian theorem to a lemma asserting the existence of a suitable set of biorthogonal functions. While in Wiener's proof this set was relatively easy to construct, in the general case it is quite difficult.

In Chapter XI it is shown that for more general kernels than those satisfying the conditions of Chapter X there are still Tauberian theorems in which the size of the a_n is not restricted, provided that the gaps in the sequence $\{\lambda_n\}$ are large enough; for the kernel $e^{-y^2/2}$ the condition is essentially $\sum 1/\lambda_n = \infty$. In Chapter XII it is shown that for this kernel the original condition $\lambda_{n+1} - \lambda_n \geqq L > 0$ still ensures the existence of a Tauberian theorem if the a_n are restricted in size (though only rather mildly compared with the restrictions required without the gap condition on the λ_n). It is also indicated how the method used would apply to more general kernels.

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