ON THE MAPPING OF QUADRATIC FORMS[1]

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The development of this paper was suggested by a theorem proposed by Bliss, proved by Albert,[2] by Reid,[3] and generalized by Hestenes and McShane.[4] That theorem had to do with two quadratic forms $P(z)$ and $Q(z)$ in real variables $z^1, z^2, \cdots, z^n$ with real coefficients, and may be stated as follows:

If $P(z)$ is positive at each point $z \neq (0)$ at which $Q(z)=0$, then there is a real number μ such that the quadratic form $P(z)+\mu Q(z)$ is positive definite.[5]

If one considers the set of points $\mathfrak{M}$ in the xy-plane into which the z-space is mapped by the transformation

$$(1) \qquad x = P(z), \qquad y = Q(z),$$

he will note that the above theorem may be interpreted as asserting the existence of a *supporting line* of the map $\mathfrak{M}$ which has contact with $\mathfrak{M}$ only at $(x, y)=(0, 0)$. This suggests that the theorem is related to the theory of convex sets.

In the present paper it is proven (Theorem 1) that $\mathfrak{M}$ is a *convex* set. Furthermore it is proven (Theorem 2) that if $P(z)$ and $Q(z)$ have no common zero except $z=(0)$, then $\mathfrak{M}$ is *closed*, and is either the entire xy-plane or an angular sector of angle less than π. Immediate corollaries include not only the theorem quoted above, but also statements of criteria for the existence of (1) semi-definite, and (2) definite linear combinations $\lambda P(z)+\mu Q(z)$. The author hopes in a subsequent paper to obtain analogous results for the general case of m quadratic forms.

Throughout the paper it is to be understood without further statement that $P(z)$ and $Q(z)$ are quadratic forms in $z^1, z^2, \cdots, z^n$, with real coefficients, the variables z^i being restricted to real values.

1. **Convexity, and the condition for $\lambda P(z)+\mu Q(z) \geqq 0$.** We give first the following theorem.

[1] Presented to the Society, December 31, 1940.

[2] This Bulletin, vol. 44 (1938), p. 250.

[3] This Bulletin, vol. 44 (1938), p. 437.

[4] Transactions of this Society, vol. 47 (1940), p. 501.

[5] While the present paper was in press, Professor N. H. McCoy kindly called the author's attention to the fact that this theorem was proven first by Paul Finsler: *Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen*, Commentarii Mathematici Helvetici, vol. 9 (1937), pp. 188–192. Apparently this work had been overlooked by the authors referred to above.

THEOREM 1. *Under the transformation* (1), *the map* $\mathfrak{M}$ *of the z-space onto the xy-plane is convex.*

If A is a point of the map, distinct from the origin O, every point of the ray OA belongs to the map, since $P(rz)=r^2P(z)$ and $Q(rz)=r^2Q(z)$ for every real number r. Hence, if A and B are two points collinear with O, and each belongs to $\mathfrak{M}$, then all points of the line segment AB belong to $\mathfrak{M}$.

We will therefore assume that $A(x_1, y_1)$ and $B(x_2, y_2)$ are points of $\mathfrak{M}$, not collinear with the origin, defined by

$$(2)\qquad \begin{aligned} x_1 &= P(z_1), & x_2 &= P(z_2), \\ y_1 &= Q(z_1), & y_2 &= Q(z_2), & z_i &= (\overset{1}{z_i}, \overset{2}{z_i}, \cdots, \overset{n}{z_i}), \end{aligned}$$

and attempt to show that every point on the line segment AB belongs to $\mathfrak{M}$. Without loss of generality we will further assume that

$$(3)\qquad x_2y_1 - x_1y_2 = k^2 > 0.$$

It will suffice to show analytically that if $\bar{t}$ is any *given* number such that $0<\bar{t}<1$, then the equations

$$(4)\qquad P(z) = x_1 + \bar{t}(x_2 - x_1), \qquad Q(z) = y_1 + \bar{t}(y_2 - y_1)$$

admit a real simultaneous solution $z=(z^1, z^2, \cdots, z^n)$.

In (4) we make the substitution

$$(5)\qquad z = \rho(z_1 \cos\theta + z_2 \sin\theta)$$

where ρ and θ are real variables, and write the results in the form

$$(6)\qquad \begin{aligned} \rho^2 p(\cos\theta, \sin\theta) &= x_1 + \bar{t}(x_2 - x_1), \\ \rho^2 q(\cos\theta, \sin\theta) &= y_1 + \bar{t}(y_2 - y_1), \end{aligned}$$

where p and q are quadratic forms in $\cos\theta$, $\sin\theta$, defined by

$$(7)\qquad \begin{aligned} p(\cos\theta, \sin\theta) &\equiv P(z_1 \cos\theta + z_2 \sin\theta), \\ q(\cos\theta, \sin\theta) &\equiv Q(z_1 \cos\theta + z_2 \sin\theta). \end{aligned}$$

Elimination of ρ^2 from the two equations (6) imposes upon θ the condition

$$(8)\qquad y_1 p(\cos\theta, \sin\theta) - x_1 q(\cos\theta, \sin\theta) = \bar{t}T(\theta)$$

where

$$(9)\qquad T(\theta) \equiv (y_1 - y_2)p(\cos\theta, \sin\theta) - (x_1 - x_2)q(\cos\theta, \sin\theta).$$

The function $T(\theta)$ is a quadratic form in $\cos\theta$, $\sin\theta$, which has the positive value k^2 at $\theta=-\pi/2$, $\theta=0$, and $\theta=\pi/2$; as can be easily verified from (7), (2), and (9). Since it can vanish for at most two values of θ between $-\pi/2$ and $\pi/2$, and must be negative between any two such values if they exist, the function $T(\theta)$ will be positive on at least one of the two intervals $-\pi/2 \leqq \theta \leqq 0$ or $0 \leqq \theta \leqq \pi/2$. We will suppose, for definiteness, that it is the latter, the argument being similar in the two cases.

We define a function $f(\theta)$ by the formula

$$f(\theta) = \frac{y_1 p(\cos\theta, \sin\theta) - x_1 q(\cos\theta, \sin\theta)}{T(\theta)}, \qquad 0 \leqq \theta \leqq \pi/2,$$

which is obviously continuous on the range indicated, and which has the further properties $f(0)=0$ and $f(\pi/2)=1$. Hence it takes on all values between 0 and 1, and in particular there is a value $\bar{\theta}$ such that $f(\bar{\theta})=\bar{t}$. This $\bar{\theta}$ is then a solution of (8).

The compatibility condition (8) being satisfied by $\theta=\bar{\theta}$, we easily satisfy the two equations (6) by taking $\rho^2=\bar{\rho}^2=k^2/T(\bar{\theta})$. And the resulting

$$z = \bar{z} = \bar{\rho}(z_1 \cos\bar{\theta} + z_2 \sin\bar{\theta})$$

given by (5) provides the required solution for (4).

COROLLARY. *A necessary and sufficient condition that there exist real* λ, μ, *such that for all real* z

$$\lambda P(z) + \mu Q(z) \geqq 0$$

is that there exist real a, b, *such that the two equations* $P(z)=a$, $Q(z)=b$ *are inconsistent for real* z.

The condition is necessary, since in its absence the map $\mathfrak{M}$ is the entire xy-plane, and every line $\lambda x+\mu y=0$ *separates* the plane into a positive half-plane and a negative half-plane, each of which contains points determined by $x=P(z)$, $y=Q(z)$.

However, if the point (a, b) does not belong to the map, no point on the ray from the origin to (a, b) belongs to the map. Hence the origin is a boundary point of the convex set $\mathfrak{M}$, and through this boundary point there passes a *supporting line* $\lambda x+\mu y=0$, such that $\lambda P(z)+\mu Q(z)\geqq 0$ for all real z.

2. **Closure, and the conditions for** $\lambda P(z)+\mu Q(z)>0$. We now prove the following theorem.

THEOREM 2. *If* $P(z)$ *and* $Q(z)$ *have no common zero except* $z=(0)$, *then* $\mathfrak{M}$ *is closed as well as convex, and is either the entire xy-plane or an angular sector of angle less than* π.

Since $\mathfrak{M}$ is convex, if it is not the entire xy-plane it lies entirely in some half-plane

$$(10) \qquad ax + by \geqq 0, \qquad a^2 + b^2 = 1.$$

We first show that, under the stated hypothesis, $\mathfrak{M}$ cannot contain both rays of the boundary line $ax+by=0$. Suppose it did contain the two symmetrical points $A(b, -a)$, $B(-b, a)$, and more explicitly that

$$P(z_1) = b, \quad Q(z_1) = -a, \quad P(z_2) = -b, \quad Q(z_2) = a.$$

Since either a or b is certainly different from zero, we may assume the notation so chosen that $a>0$. Then $Q(z_1)<0$ and $Q(z_2)>0$. Hence[6] there are, in the hyperplane defined by $z=z_1u+z_2v$, two linearly independent points $z_0=z_1u_0+z_2v_0$, $z_0'=z_1u_0'+z_2v_0'$, such that

$$(11) \qquad Q(z_0) = Q(z_0') = 0.$$

Consider now the quadratic form

$$\phi(u, v) = aP(z_1u + z_2v) + bQ(z_1u + z_2v)$$

in the two real variables u, v. It is easily verified that ϕ vanishes at $(u, v)=(1, 0)$ and at $(u, v)=(0, 1)$. These, together with the dependent points $(c, 0)$ and $(0, c)$, are its only possible zeros unless it vanishes identically. It does not vanish identically, since it does not vanish at (u_0, v_0) or (u_0', v_0') in view of (11) and our hypothesis. Hence, by (10), $\phi(u, v)>0$ except at $(c, 0)$ and $(0, c)$. This is clearly impossible, and the contradiction proves that the map $\mathfrak{M}$ cannot contain both points $A(b, -a)$ and $B(-b, a)$.

We now let $X(x, y)$ denote any point of $\mathfrak{M}$, and consider the angle AOX, where $A\equiv A(b, -a)$ and $O\equiv O(0, 0)$. Then $\cos AOX = (bx-ay)/(x^2+y^2)^{1/2}$. And as the point z varies over the unit hypersphere $\|z\|=1$, $\cos AOX$ is represented by the function

$$\psi(z) = \frac{bP(z) - aQ(z)}{[P^2(z) + Q^2(z)]^{1/2}}, \qquad \|z\| = 1.$$

In view of the hypothesis, $\psi(z)$ is continuous on this hypersphere; and since its values are bounded below by -1 and above by $+1$, it attains a minimum value $m\geqq -1$ and a maximum value $M\leqq 1$. It is impossible that $m=-1$ *and* $M=1$, since then the map $\mathfrak{M}$ would contain both points $A(b, -a)$ and $B(-b, a)$. Hence $\mathfrak{M}$ consists of a closed

[6] Reference may be made to Bôcher, *Introduction to Higher Algebra*, p. 151, Theorem 2.

sector bounded by rays OA' and OB' such that $\cos AOA' = M$ and $\cos AOB' = m$. And *angle* $A'OB' <$ *angle* $AOB = \pi$.

COROLLARY 1. *Necessary and sufficient conditions that there exist real* λ, μ, *such that for all real* $z \neq (0)$

$$\lambda P(z) + \mu Q(z) > 0 \tag{12}$$

are that: (1) *there exist real* a, b, *such that the two equations* $P(z) = a$, $Q(z) = b$ *are inconsistent for real* z; *and* (2) $P(z)$ *and* $Q(z)$ *have no common zero except* $z = (0)$.

The necessity is obvious. The sufficiency follows from Theorem 2. For if $(\lambda, \mu) \neq (0, 0)$ is a point of $\mathfrak{M}$ on the bisector of its angular sector, then (12) is satisfied.

COROLLARY 2. (Bliss-Albert theorem.) *If, whenever* $Q(z) = 0$ *and* $z \neq (0)$, $P(z) > 0$; *then there exists a real number* μ *such that* $P(z) + \mu Q(z)$ *is positive definite.*

The conditions of Corollary 1 are obviously satisfied with $(a, b) = (-1, 0)$. Hence there exist λ, μ, satisfying (12). If $Q(z)$ actually vanishes for some $z \neq (0)$, λ is necessarily positive and hence may be taken equal to 1.

If, on the contrary, $Q(z)$ is definite, then the map $\mathfrak{M}$ is a *closed* sector of which only the vertex (0, 0) is on the x-axis. Hence there is a line $x + \mu y = 0$ such that $x + \mu y > 0$ for all points of $\mathfrak{M}$ except (0, 0). Then $P(z) + \mu Q(z)$ is positive definite.

It is perhaps worthy of note that the two conditions of Corollary 1 are completely independent. This is shown by the following four examples.

Example 1, in which both (1) and (2) are satisfied:

$$P(u, v) \equiv u^2, \qquad Q(u, v) \equiv v^2.$$

Example 2, in which (1) is satisfied but (2) is not:

$$P(u, v) \equiv u^2, \qquad Q(u, v) \equiv uv.$$

Example 3, in which (1) is not satisfied but (2) is:

$$P(u, v) \equiv u^2 + 2uv, \qquad Q(u, v) \equiv 2uv + v^2.$$

Example 4, in which neither (1) nor (2) is satisfied:

$$P(u, v, w, t) \equiv u^2 + 2uv + w^2, \qquad Q(u, v, w, t) \equiv 2uv + v^2 + wt.$$

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