

A NOTE ON QUASI-METRIC SPACES¹

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1. **Introduction.** In a recent abstract² the author introduced an important application of a class of neighborhood spaces in which pairs of points are required to satisfy only a very weak separation axiom (due to Kolmogoroff):

(K) *If x and y are distinct points, then at least one of these points has a neighborhood which does not contain the other.*

In the present note it is shown that a very useful generalized distance function may be defined in certain of these spaces.

Clearly, any such distance function must be an asymmetric one. W. A. Wilson³ considered the definition of asymmetric distances in certain spaces which satisfy stronger separation axioms than K. It is shown here that a slight modification of one of the axioms in [W] allows the extension of a large part of the theory developed there to spaces subject to K.

Since many of the theorems and proofs in [W] remain valid here with only very obvious changes, this note will be limited to a mere sketch concerning new properties which arise under the weaker axioms used. The reader should have no difficulty in adapting the more complete discussion given in [W] to the case studied here.

2. **The distance function.** The symbol 1 will denote a space of points; points and point sets in 1 will be denoted by small and capital letters respectively.

The space 1 will be said to be *quasi-metric* if for every pair of points x, y in 1 there are defined two non-negative numbers xy and yx , not necessarily equal, which satisfy the following postulates:

- I. $xy = yx = 0$ if and only if $x = y$;⁴
- II. for any three points, $xy \leq xz + zy$.

If for some pair of points $xy = 0 \neq yx$, then the point x will be said to be *adjacent* to the point y . It is to be emphasized that, regardless

¹ Presented to the Society, April 13, 1940.

² Abstract 46-3-138, this Bulletin.

³ W. A. Wilson, *On quasi-metric spaces*, American Journal of Mathematics, vol. 53 (1931), p. 675. Hereafter this paper will be referred to as [W].

⁴ In [W] the stronger axiom: $xy = 0$ if and only if $x = y$ was used. This excludes the case $xy = 0 \neq yx$ allowed here.

of the connotation of symmetry familiar with the word adjacent, the relationship to which it refers here is an asymmetric one.

3. Limits, closed and open sets. The nature of the function xy leads to the definition of three separate types of limit points and the corresponding classes of closed and open sets. A point p will be called a *u-limit*, *l-limit*, or a *c-limit* of a point set A (of a sequence $\{x_n\}$) if for every positive number ϵ there exists a point x in $A - p$ such that $px < \epsilon$, $xp < \epsilon$, or both simultaneously, respectively (if there is a number N such that $px_n < \epsilon$, and so on, for $n > N$).

The point set $S_u(a, r)$ consisting of all points x such that $ax < r$ will be called a *u-sphere* of center a and radius r . Similarly one defines the spheres $S_l(a, r)$ and $S_c(a, r)$ by the relations $xa < r$ and both $ax < r$ and $xa < r$.

A point set will be termed *u-open*, *l-open*, or *c-open* if for every point p in the set some sphere S_u , S_l , or S_c , respectively, about p is contained entirely within the set. Such sets will be denoted by G_u , G_l , and G_c respectively. A point set will be termed *u-closed*, *l-closed*, or *c-closed* if it contains all of its limit points of the respective class. These sets will be denoted by F_u , F_l , and F_c respectively.

The space 1 will be called *separable* if there exists a denumerable sequence of points which is everywhere dense in 1 in the sense of *c-limits*.

Just as in the theory of ordinary topological spaces, one proves that the complement of a set G_u (G_l or G_c) is a set F_u (F_l or F_c). It is evident that every set F_u or F_l is also a set F_c and that every set G_u or G_l is also a set G_c ; however the converses are not true.

It is to be noted that a point p may be a *u-limit* and at the same time an *l-limit* of a point set A without being a *c-limit* of that set. On the other hand, for sequences one has the following theorem:

THEOREM 1. *If $x = u\text{-lim } x_n$ and $x = l\text{-lim } x_n$ then $x = c\text{-lim } x_n$. Moreover, there exists no other *c-limit* of the sequence $\{x_n\}$ and the Cauchy criterion of convergence is satisfied by the sequence.*

PROOF. The first clause is immediate from the definitions.

Suppose that x and y are both *c-limits* of the sequence $\{x_n\}$. It follows that for an arbitrary positive number ϵ there exist numbers n' and n'' such that $xy \leq xx_n + x_ny < \epsilon$ for $n > n'$ and $yx \leq yx_n + x_nx < \epsilon$ for $n > n''$. Thus $xy = 0 = yx$ and by I, $x = y$.

Again, $x = c\text{-lim } x_n$ implies that for arbitrary $\epsilon > 0$ there is an n' such that $x_nx_m \leq x_nx + xx_m < \epsilon$ for $n > n'$ and the Cauchy criterion is satisfied.

The following theorems indicate the relations between the several limits of a sequence and the property of adjacency.

THEOREM 2. *If $x = l\text{-lim } x_n$ and $y = u\text{-lim } x_n$ for the same sequence $\{x_n\}$, then either $x = y$ or else y is adjacent to x .*

The proof is immediate from $yx \leq yx_n + x_nx$ and the fact that $yx_n \rightarrow 0$ and $x_nx \rightarrow 0$.

COROLLARY. *If $x = c\text{-lim } x_n$ and $y = u\text{-lim } x_n$ then either $y = x$ or y is adjacent to x ; if $y = l\text{-lim } x_n$, then either $y = x$ or x is adjacent to y .*

It is worthy of note that if a point x is adjacent to a point y then x is a u -limit of the point set consisting of the single point y . Thus, if the usual definition of connected set be adopted with limits taken in the u -sense, the point set $(x+y)$ forms a connected set.

4. Topological versus quasi-metric spaces. Hausdorff⁵ stated the following axioms for topological spaces:

- (A) Every point p has a neighborhood U_p . For every U_p , $p \in U_p$.
- (B) For every two neighborhoods U_p and V_p of the point p , there is a third $W_p \subset U_p V_p$.
- (C) For every point q in U_p , there is a neighborhood $U_q \subset U_p$.
- (D) To each point p there corresponds a denumerable set of neighborhoods such that every neighborhood U_p contains a neighborhood of that set.
- (E) There exists a denumerable set of neighborhoods $\{U_n\}$ such that if U_p is any neighborhood of the point p , then there is an n such that $p \in U_n \subset U_p$.

THEOREM 3. *Let 1 denote a quasi-metric space and, for each point x in 1 , let the class of spheres $S_u(x, r)$ of rational radii r be termed neighborhoods of x . Then 1 is a topological space satisfying the axioms A, B, C, D, and K. The u -limits of sequences of points in 1 are the topological limits of the sequences and conversely. If the space is separable, Axiom D may be replaced by the stronger Axiom E. The theorem remains valid if the spheres S_u are replaced by the spheres S_l and u -limits by l -limits.*

The proof is evident.

THEOREM 4. *Let 1 be a neighborhood space satisfying the axioms A, B, C, E, and K. Then distances can be defined in 1 so that the space is a separable, quasi-metric space. A topological limit of a sequence of*

⁵ Hausdorff, *Mengenlehre*, Berlin, 1935, pp. 228, 229.

points in 1 is a u -limit of the sequence and conversely. The theorem remains valid if u -limit is replaced by l -limit provided that a corresponding change be made in the distance function.

Since the proof of this theorem is long but follows that of Theorem IV, §7, in [W] with only a few minor changes, it seems sufficient to state the definition of the distance function for the case of u -limits.

Let the denumerable system of neighborhoods postulated by Axiom E be denoted by $\{U_n\}$. For any pair of points x and y in 1 and for each $n=1, 2, 3, \dots$, define the function $f_n(x, y)$ to be equal 1 if x is in U_n and y is in $1-U_n$, or equal to 0 in case x is in $1-U_n$ or in case $x+y$ is in U_n . Then define the function

$$xy = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x, y).$$

It should be remarked that a quasi-metric is not necessarily a continuous function of either variable.

5. Applications. The distance function described above furnishes an exceedingly simple topologization for a very general type of collection of closed point sets. Let S denote a compact metric space and let $S = \sum X$ be any decomposition of S into closed point sets. It is not assumed that the sets of the decomposition are disjoint. However it is assumed that if X_1 and X_2 are distinct elements of the decomposition which overlap, then $X_1 X_2 = X$ is again an element of the decomposition. Moreover, if X_3 is any element which intersects X , then $X_1 X_3 = X = X_2 X_3$. Such a collection is afforded by the *upper semi-continuous collection of the second type* introduced by R. L. Moore⁶ and, in particular, by the so-called hyperspace of cyclic elements of a continuous curve. If, now, the elements of the collection are thought of as points of a new space 1, the topologization is introduced as follows: if X and Y are two elements of 1, the distance function $XY =$ the greatest lower bound of the numbers r such that, as seen in the space S , every point x of the set X has a distance from the set Y which is less than r . The space is easily shown to be quasi-metric in the sense described in the preceding paragraphs.

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⁶ The Rice Institute Pamphlet, vol. 23, no. 1, 1936, p. 42.