ELEMENTARY PROOF OF A THEOREM ON LORENTZ MATRICES

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Let x and y be real n- and m-vectors and x^2, y^2 the scalar squares of x, y. The corresponding Lorentz matrices are matrices of $(n+m)$-dimensional real linear transformations which leave the quadratic form x^2-y^2 invariant. Let the transformation be written in the form

$$(1)\qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}.$$

Then *the signs of the determinants* $|A|$ *and* $|D|$ *form two* 1*-dimensional representations of the Lorentz group*. Two algebraic proofs at present available for this fact[1] depend on a recursive factorization of the Lorentz matrix into simple factors or on deeper facts from the theory of representations. On the other hand, a simple topological proof may be given in quite an obvious manner. In this note the topological proof is briefly sketched and then a simple algebraic proof is given which does not depend on recursive factorization or representation theory and is valid in any real field.

The set defined by $x^2-y^2\geqq 1$ in the real $(n+m)$-dimensional space possesses one basic $(n-1)$-dimensional (finite) cycle Γ which can most easily be represented by the $(n-1)$-dimensional basic cycle of the $(n-1)$-dimensional sphere $x^2=1$, $y=0$. Now Γ is transformed by (1) into a cycle homologous to $+\Gamma$ or to $-\Gamma$ according as $|A|$ is positive or negative. The formal proofs of these topological facts are obtained most easily from the remark that the whole space $x^2-y^2\geqq 1$ can be retracted into its subset $x^2=1$, $y=0$ by a deformation which does not change the value of $x/(x^2)^{1/2}$ for any point. That *sign* $|A|$ is a one-dimensional representation of the Lorentz group is of course evident from the fact that Γ is transformed by (1) into a cycle homologous to *sign* $|A|\cdot\Gamma$. The statement concerning the signature of $|D|$ depends on a similar consideration for the set defined by $x^2-y^2\leqq -1$.

Now let the elements of the matrix in (1) belong to any real field. Let the unit matrices of dimensions n and m be denoted by E_n and E_m. The fact that the matrix in (1) is a Lorentz matrix may be expressed by the relations:

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[1] Cf. W. Givens, *Factorization and signatures of Lorentz matrices*, this Bulletin, vol. 46 (1940), pp. 81–85, where other references are given. My thanks are due to Dr. Murnaghan who drew my attention to the above theorem.

$$(2) \qquad A'A - C'C = E_n, \qquad D'D - B'B = E_m, \qquad A'B = C'D,$$

which may be obtained by forming the expression $x^2 - y^2$ for the vector on the right in (1).

If P is a matrix of m rows and n columns, such that $E_n - P'P$ is positive definite, then the sign of the determinant $|A+BP|$ is independent of P; in particular $|A+BP| \neq 0$ and $|A| \neq 0$.

In fact, from (2) one easily obtains the identity

$$(A + BP)'(A + BP) = (C + DP)'(C + DP) + E_n - P'P.$$

Since $E_n - P'P$ is assumed to be positive definite, this implies that $(A+BP)'(A+BP)$ is positive definite. Thus $|A+BP| \neq 0$ and, by choosing $P=0$, also $|A| \neq 0$. On replacing P by tP, one sees that the determinant $|A+tBP|$, which is a polynomial in the parameter t, is never 0 while $-1 \leqq t \leqq 1$. For $E_n - t^2P'P = E_n - P'P + (1-t^2)P'P$ is positive definite if $-1 \leqq t \leqq 1$. Thus the polynomial $|A+tBP|$ cannot change its sign as t varies between 0 and 1. In the field of real numbers this is evident. If the underlying field is any real field, and if the polynomial $|A+tBP|$ took both possible signs for $-1 \leqq t \leqq 1$, then one could adjoin to the field a root of $|A+tBP| = 0$ which lies between -1 and 1. In the enlarged field one obtains of course a contradiction with the fact that $|A+tBP| \neq 0$ for $-1 \leqq t \leqq 1$.

Let the product of two Lorentz matrices be written in the form

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix}.$$

Then one has

$$A_1A_2 + B_1C_2 = (A_1 + B_1C_2A_2^{-1})A_2 = (A_1 + B_1P)A_2,$$

where $P = C_2A_2^{-1}$. But

$$\begin{aligned} E_n - P'P &= E_n - (A_2')^{-1}C_2'C_2A_2^{-1} = (A_2')^{-1}(A_2'A_2 - C_2'C_2)A_2^{-1} \\ &= (A_2')^{-1}A_2^{-1} \end{aligned}$$

is a positive definite matrix, so that sign $|A_1+B_1C_2A_2^{-1}| =$ sign $|A_1+B_1P| =$ sign $|A_1|$. Thus

$$\operatorname{sign} |A_1A_2 + B_1C_2| = \operatorname{sign} |A_1| \cdot \operatorname{sign} |A_2|.$$

This completes the algebraic proof of the above theorem.

The geometrical content of the proof becomes clearer, if one realizes that the n-dimensional linear spaces with the equations $y = Px$ (where $E_n - P'P$ is positive definite) are precisely those spaces through the origin which meet the quadratic $x^2 - y^2 = 1$ in a completely elliptical quadratic (and the cone $x^2-y^2=0$ in its vertex only). Thus this system of linear spaces is invariant under the Lorentz group. That the sign of $|A+BP|$ is independent of P means that the orientation of all spaces $y=Px$ is left invariant by a Lorentz matrix with $|A|>0$ and is changed into its opposite by a Lorentz matrix with $|A|<0$. Complications in preceding proofs of the theorem apparently originate either from the inclusion of the proof that every matrix P with positive definite $E-P'P$ is the matrix CA^{-1} of a Lorentz transformation (1) and/or of the proof that the subgroup of the Lorentz group defined by $|A|>0$, $|D|>0$ is connected.

The 1-*dimensional representation of the Lorentz group given by the determinants of the Lorentz matrix* (1) *is the product of the two representations given by the signs of* $|A|$ *and* $|D|$. In fact, D as well as A is nonsingular. Thus[2]

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \cdot \begin{vmatrix} A & B \\ D^{-1}C & E_m \end{vmatrix} = |D| \cdot \begin{vmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & E_m \end{vmatrix}$$
$$= |D| \cdot |A - BD^{-1}C|,$$

so that, since $BD^{-1}=A'^{-1}C'$ and $|A'|=|A|$,

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \cdot |A - A'^{-1}C'C| = \frac{|D|}{|A|}|A'A - C'C| = \frac{|D|}{|A|}.$$

Thus the sign of

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

is the product of the signs of $|D|$ and $|A|$. Since

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \frac{|A|}{|D|}$$

may be similarly proved, one has

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \text{sign } |A| \cdot \text{sign } |D|.$$

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[2] Cf. J. Williamson, *The expansion of determinants of composite order*, American Mathematical Monthly, vol. 40 (1933), pp. 65–69.