PROOF OF A THEOREM OF HALL

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In the Journal of the London Mathematical Society for July, 1937, Mr. Philip Hall gave a proof of the theorem, "If a group G of order g has a subgroup of order m for every divisor m of g such that $(m, g/m)=1$, then G is a soluble group." The proof is a very simple one in contrast to the rather difficult proof of the converse theorem which Hall had published in the same journal for April, 1927. It seems worthwhile to give a simpler proof of this converse.

THEOREM. *If g is the order of a soluble group G and m is a divisor of g such that m and g/m are relatively prime, then G has a subgroup of order m and furthermore all the subgroups of order m in G are conjugate under G.*

PROOF. Since the theorem is true by default for prime power groups, let us suppose that it is true for all soluble groups of orders less than g and use the method of complete induction. Then the theorem is true for an invariant subgroup G' of prime index r in G.

If r divides g/m, then G' has a subgroup of order m. Since G' is invariant and $(m, r)=1$, every element of order dividing m in G must be in G'. Hence all the subgroups of order m in G must be in G' where by hypothesis they form a complete set of conjugates under G' and consequently a complete set of conjugates under G.

If r divides m, then G' has a complete set of conjugates of order m/r. Since G' is invariant, the subgroups of order m/r in G' are a complete set of conjugates under G. Let M' be one of these subgroups of order m/r in G'.

If M' is the only subgroup of order m/r in G', then, it must be invariant in G. Then the quotient group is of order rg/m and since r is a Sylow divisor of the order of this quotient group, it has a subgroup of order r. Then G has a subgroup of order $r\cdot m/r$ or m. On the other hand if M' is not the only subgroup of order m/r in G', let it be one of k subgroups of order m/r in G'. Then k divides the order g/r of G' and the normalizer of M' in G is of order g/k divisible by m. Since this normalizer is a soluble group of order g/k less than g, it has a subgroup of order m.

There remains only to show, for the case r divides m, that all the subgroups of order m in G are conjugate under G. Let M be a subgroup of order m. If there is no other subgroup of order m in G, then the theorem is true by default. However, if M_1 is another subgroup of order m in G it will be shown that M and M_1 are conjugate under G.

Let the crosscut of M and G' be Γ of order γ and the crosscut of M_1 and G' be Γ_1 of order γ_1. Then since G is generated by M and G' as well as by M_1 and G' and since G' is invariant under both M and M_1 we have

$$(g/r)(m)/\gamma = g, \qquad (g/r)(m)/\gamma_1 = g,$$

whence

$$\gamma = \gamma_1 = m/r.$$

Then since Γ and Γ_1 are of order m/r, they are conjugate under G'. If $S^{-1}\Gamma_1 S = \Gamma$, then $S^{-1}M_1S$ and M have a common invariant subgroup Γ. If Γ is invariant in G, the quotient group G/Γ is of order gr/m. Since r is a Sylow divisor, the subgroups M/Γ and $S^{-1}M_1S/\Gamma$ of order r are conjugate under G/Γ and hence M and $S^{-1}M_1S$ are conjugate under G as was to be proved. If, however, Γ is not invariant under G, its normalizer is a proper subgroup of G containing M and $S^{-1}M_1S$ which are therefore conjugate under the normalizer of Γ as was to be proved.

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