A NOTE ON A THEOREM OF RADÓ CONCERNING THE $(1, m)$ CONFORMAL MAPS OF A MULTIPLY-CONNECTED REGION INTO ITSELF[1]

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Let G_z denote a region in the z-plane and let $w=f(z)$ be a function of z defined for $z \subset G_z$ which has the following properties: (1) $w=f(z)$ is analytic and single-valued for $z \subset G_z$, (2) $z \subset G_z$ implies that $f(z) \subset G_z$, (3) to each point $w_0 \subset G_z$ there correspond m and only m points $z_0^{(k)}$ $(k=1, 2, \cdots, m)$ contained in G_z such that $f(z_0^{(k)})=w_0$ $(k=1, 2, \cdots, m)$ where following the usual convention we count the $z_0^{(k)}$ according to their multiplicities. Then $w=f(z)$ is said to define a $(1, m)$ *conformal map of G_z onto itself*. Such maps have been studied by Fatou[2] and Julia[3] for the case where G_z is simply-connected, and by Radó[4] who treated multiply-connected regions as well. Among the results which Radó established is the following theorem:

Let G_z be a region of finite connectivity p (>1); then there exists no $(1, m)$ conformal map of G_z onto itself for $m>1$.

Let us remark with Radó that the theorem is no longer valid if G_z is of infinite connectivity, as simple examples from the theory of the iteration of rational functions show.[5] Radó's proof of the theorem just cited is based on the possibility of mapping one-to-one and conformally a region of finite connectivity p, none of the components of its boundary reducing to points, onto a region of connectivity p, the boundary of which consists of p disjoint circles. Other types of canonical regions yield the same result, notably one due to Koebe.[6] It is the object of the present note to establish Radó's theorem directly without appeal to the possibility of mapping one-to-one and conformally the region G_z onto a canonical region. Our tools are the theory of iteration and a simple modification of Nevanlinna's principle of harmonic measure.[7]

Let G_z, the region we are going to study, have as its boundary p (>1) disjoint continua Γ_k $(k=1, 2, \cdots, p)$. It is evident that we

[1] Presented to the Society, April 27, 1940, under the title *A note on a theorem of Radó*.

[2] P. Fatou, Bulletin de la Société Mathématique de France, 1919.

[3] G. Julia, Comptes Rendus de l'Académie des Sciences, Paris, vol. 166 (1918).

[4] T. Radó, Acta Szeged, vol. 1 (1922).

[5] G. Julia, Journal de Mathématiques Pures et Appliquées, 1918.

[6] P. Koebe, Acta Mathematica, vol. 41.

[7] R. Nevanlinna, *Eindeutige analytische Funktionen*, chap. 3.

may assume without loss of generality that the Γ_k are all closed Jordan curves. Further let $w=f(z)$ define a $(1, m)$ conformal map of G_z onto itself. In Radó's paper cited above, it is shown that if $w=f(z)$ defines a $(1, m)$ conformal map of G_z onto itself, then whenever z tends to the boundary of G_z, so does $f(z)$, in such a manner that whenever z tends to a given component Γ_k of the boundary, then $f(z)$ tends to one and only one component of the boundary Γ_{l_k} where the index l_k depends on k. Now the relation $k \to l_k$ $(k=1, 2, \cdots, p)$ defines a permutation of the indices $1, 2, \cdots, p$.

We shall understand by $f_n(z)$, *the nth iterate of* $f(z)$, that function defined by the recursive relations

$$\text{(A)} \qquad f_0(z) \equiv z, \; f_1(z) \equiv f(z), \cdots, f_n(z) \equiv f[f_{n-1}(z)].$$

By the definition of $w=f(z)$ it is clear that the definition given by (A) for $f_n(z)$ is meaningful. Since the relation $k \to l_k$ defines a permutation of the indices $1, 2, \cdots, p$, a suitably chosen power of this permutation is the identity. Hence for a properly chosen whole number n_0, $f_{n_0}(z)$ has the property that when z tends to a component Γ_k of the boundary of G_z, then $f_{n_0}(z)$ tends to exactly the same component Γ_k. Let us denote $f_{n_0}(z)$ by $F(z)$.

We are now in a position to demonstrate Radó's theorem directly. By the *harmonic measure of z with respect to* Γ_k, denoted by $\omega(z, \Gamma_k, G_z)$ we understand that harmonic function defined for $z \subset G_z$ which is single-valued and bounded for $z \subset G_z$ and further takes on the boundary value 1 on Γ_k and the boundary value 0 on all the components of the boundary of G_z exclusive of Γ_k.[8]

The relations

$$\text{(B)} \qquad \omega(F(z), \Gamma_k, G_z) \equiv \omega(z, \Gamma_k, G_z), \qquad k = 1, 2, \cdots, p,$$

are an immediate consequence of the principle of the maximum for harmonic functions. For, as z tends to a given component of the boundary of G_z, $F(z)$ tends to the same component. Hence $\omega(F(z), \Gamma_k, G_z)$ has the same boundary values as $\omega(z, \Gamma_k, G_z)$, hence the identity. To establish Radó's theorem we need consider only one of these identities, say

$$\text{(C)} \qquad \omega(F(z), \Gamma_1, G_z) \equiv \omega(z, \Gamma_1, G_z).$$

Let z_0 be a point lying in G_z, and let $\omega(z_0, \Gamma_1, G_z) = \lambda_0$ $(0 < \lambda_0 < 1)$. Then our identity (C) implies

$$\omega(F_n(z_0), \Gamma_1, G_z) = \lambda_0,$$

[8] R. Nevanlinna, ibid.

where $F_n(z)$ is the nth iterate of $F(z)$ for $n=1, 2, \cdots$. Hence if z is on a given level curve defined by $\omega(z, \Gamma_1, G_z)=\lambda$ $(0<\lambda<1)$, $F_n(z)$ is on the same level curve for $n=1, 2, \cdots$. This permits us to conclude that no limit function of the sequence $\{F_n(z)\}$ is a constant. From this fact we infer that $w=F(z)$ defines a (1, 1) conformal map of G_z onto itself, and hence so does $w=f(z)$; that is, *m cannot exceed one.*

Suppose contrary to our assertion that $m>1$. Then $w=F(z)$ would define a $(1, m^{n_0})$ conformal map of G_z onto itself and $w=F_n(z)$ would define a $(1, m^{nn_0})$ conformal map of G_z onto itself. Let $\{F_{k_n}(z)\}$ be a convergent subsequence of $\{F_n(z)\}$ and let $F_0(z)$ denote the limit function of this subsequence. A point w_0 has, under the map defined by $w=F_{k_n}(z)$, $m^{k_n n_0}$ antecedents all of which lie on the level curve defined by $\omega(z, \Gamma_1, G_z)=\omega(w_0, \Gamma_1, G_z)$. But by Hurwitz's theorem, for k_n sufficiently large $F_{k_n}(z)-w_0$ and $F_0(z)-w_0$ have the same number of zeros, and this is manifestly impossible. Hence m cannot exceed one.

It is interesting to note that the technique employed in the present proof of Radó's theorem, that is, a combined use of the theory of iteration and of an extended form of the principle of harmonic measure (the modification consisting in the fact that we did not require the continuity of $F(z)$ on Γ_1) may be applied to other problems in the theory of the conformal mapping of multiply-connected regions. In particular, such a technique permits us to conclude that the number of (1, 1) conformal maps of a given multiply-connected region of finite connectivity p, where $p>2$, bounded by p disjoint continua is finite. Koebe's proof of this theorem is based on the fact that the region in question can be mapped one-to-one and conformally onto a canonical region. We shall not give in the present note the details of a direct proof of this theorem. Let us remark however that it is based on the fact that for $p>2$ each of the harmonic functions $\omega(z, \Gamma_k, G_z)$ $(k=1, 2, \cdots, p)$ has precisely $p-2$ (>0) critical points. A study of the level curves containing these critical points yields the desired result.

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