MAXIMUM OF CERTAIN FUNDAMENTAL LAGRANGE INTERPOLATION POLYNOMIALS[1]

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This note extends some of the results obtained in a previous paper[2] which we shall designate as I. The notations are the same.

We are concerned with the polynomials

$$l_k^{(n)}(x) \equiv \frac{\phi_n(x)}{\phi_n'(x_k)(x - x_k)}, \qquad k = 1, 2, \cdots, n,$$

where $\phi_n(x) \equiv (x-x_1)(x-x_2) \cdots (x-x_n)$ is the Jacobi polynomial of degree n which satisfies the differential equation $(1-x^2)\phi_n''(x) + [\alpha-\beta-(\alpha+\beta)x]\phi_n'(x)+n(n+\alpha+\beta-1)\phi_n(x)=0$. The parameters α, β are positive and n is a positive integer. It is known that $-1<x_n<x_{n-1}< \cdots <x_1<1$. Throughout the paper, x is always restricted to the interval $-1 \leqq x \leqq 1$.

It was shown in I, for example, that, if $\alpha=\beta=\frac{3}{2}$, max $|l_k^{(n)}(x)| <2$ and $l_1^{(n)}(1) \to 2$ as $n \to \infty$.

Now we use[3]

$$\phi_n(1) = \frac{2^n\Gamma(n+\beta)\Gamma(n+\alpha+\beta-1)}{\Gamma(\beta)\Gamma(2n+\alpha+\beta-1)}$$

and the asymptotic expressions[4]

$$\phi_n(\cos\theta) = \frac{2^n\Gamma(n+1)\Gamma(n+\alpha+\beta-1)}{(\pi n)^{1/2}\Gamma(2n+\alpha+\beta-1)}\left(\sin\frac{\theta}{2}\right)^{1/2-\beta}\left(\cos\frac{\theta}{2}\right)^{1/2-\alpha} \cdot\left\{\cos\left[N\theta-(2\beta-1)\pi/4\right]+(n\sin\theta)^{-1}O(1)\right\},$$

$$\phi_n(\cos\theta) = \frac{2^n\Gamma(n+1)\Gamma(n+\alpha+\beta-1)}{\Gamma(2n+\alpha+\beta-1)}\left(\sin\frac{\theta}{2}\right)^{1-\beta}\left(\cos\frac{\theta}{2}\right)^{1-\alpha} \cdot\left\{\frac{\Gamma(n+\beta)}{\Gamma(n+1)}\left(\frac{\theta}{\sin\theta}\right)^{1/2}\frac{J_{\beta-1}(N\theta)}{N^{\beta-1}}+\theta^{1/2}O(n^{-3/2})\right\},$$

where $N=n+(\alpha+\beta-1)/2$, $cn^{-1} \leqq \theta \leqq \pi-\epsilon$, c, ϵ positive constants and

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[1] Presented to the Society, April 13, 1940.

[2] M. Webster, *Note on certain Lagrange interpolation polynomials*, this Bulletin, vol. 45 (1939), pp. 870–873.

[3] C. Winston, *On mechanical quadratures formulae involving the classical orthogonal polynomials*, Annals of Mathematics, (2), vol. 35 (1934), pp. 658–677.

[4] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939, pp. 191–192, 121, 123.

where $J_m(x)$ is Bessel's function of order m. Since $\phi_n'(x; \alpha, \beta) = n\phi_{n-1}(x; \alpha+1, \beta+1)$, these yield immediately the following results:

LEMMA. *For* x_k *such that* $-1+\epsilon \leqq x_k \leqq 1-\epsilon$ *and* $|x-x_k| \geqq \epsilon' > 0$, $\max |l_k^{(n)}(x)| \to 0$ as $n \to \infty$ *even if* $x \to \pm 1$ $(\alpha, \beta < \frac{3}{2};\ \epsilon, \epsilon' > 0)$.

THEOREM 1. *For* x_k *such that* $-1+\epsilon \leqq x_k \leqq 1-\epsilon$ *and* $|x-x_k| \geqq \epsilon' > 0$, $\max |l_k^{(n)}(x)| = O(n^\gamma)$ *as* $n \to \infty$ *where* $\max(\alpha, \beta) = \gamma > \frac{3}{2};\ \epsilon, \epsilon' > 0$. *The exponent* γ *cannot be decreased.*

The method used in the proof of Theorem 5 in I really gives the following slightly stronger result:

THEOREM 2. *If* $-1+\epsilon \leqq x_k \leqq 1-\epsilon$, $-1+\epsilon' \leqq x \leqq 1-\epsilon'$, $\max |l_k^{(n)}(x)| \to 1$ *as* $n \to \infty$ $(\epsilon, \epsilon' > 0)$.

Combining Theorem 2 and the lemma, we obtain the following:

THEOREM 3. *For* x_k *such that* $-1+\epsilon \leqq x_k \leqq 1-\epsilon$, $\max |l_k^{(n)}(x)| \to 1$ *as* $n \to \infty$ $(\alpha, \beta < \frac{3}{2}, \epsilon > 0)$.

This result is a considerable improvement over Theorem 5 in I. Moreover, if the hypothesis $-1+\epsilon \leqq x_k \leqq 1-\epsilon$ is removed, the theorem is not true as Erdös and Grünwald[5] showed in case $\alpha = \beta = \frac{1}{2}$. In view of Theorems 1, 4, 5, 6, the restriction $\alpha, \beta < \frac{3}{2}$ is also necessary. In particular, this theorem holds for the case of Tschebycheff $(\alpha = \beta = \frac{1}{2})$ and Legendre $(\alpha = \beta = 1)$ polynomials.

THEOREM 4. *If* $\alpha = \beta = \frac{3}{2}$ *and* $x_k \to t$ *as* $n \to \infty$, *then* $\max |l_k^{(n)}(x)| \to 1 + |t|$ *as* $n \to \infty$ $(-1 \leqq t \leqq 1)$. *This is also an upper bound if* $|x_k| < |t|$ *at least for large values of* n.

PROOF. It was shown in I that $l_k^{(n)}(1) = 1 + x_k$ and for $x_{k+1} \leqq x \leqq x_{k-1}$, $\max |l_k^{(n)}(x)| < 1.87$. Since (I) $\max |l(_k^{(n)}x)|$ is attained either between x_{k+1} and x_{k-1} or at $x = \pm 1$, the theorem is valid for $t = 1$ and, by symmetry, for $t = -1$.

If $|t| < 1$, the preceding paragraph and Theorem 2 complete the proof. In fact, $\max |l_k^{(n)}(x)| = 1 + |x_k|$ at least for large n.

The next two theorems are obtained in a similar manner.

THEOREM 5. *If* $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$ *and* $x_k \to t$ as $n \to \infty$, *then* $\max |l_k^{(n)}(x)| \to 4/\pi$ *if* $t = -1$, 1 if $-1 < t \leqq -\frac{1}{2}$, $(2(1+t))^{1/2}$ *if* $-\frac{1}{2} \leqq t \leqq 1$.

THEOREM 6. *If* $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$ *and* $x_k \to t$ *as* $n \to \infty$, *then* $\max |l_k^{(n)}(x)| \to [2(1-t)]^{1/2}$ *if* $-1 \leqq t \leqq \frac{1}{2}$, 1 *if* $\frac{1}{2} \leqq t < 1$, $4/\pi$ *if* $t = 1$.

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[5] Erdös and Grünwald, *Note on an elementary problem of interpolation*, this Bulletin, vol. 44 (1938), pp. 515–518.

The max $|l_1^{(n)}(x)|$ is attained at $x=\pm 1$ since[4] (I) $\theta_{k+1}-\theta_k \leqq 2\pi/(2n+\alpha+\beta-1)$ provided $\frac{1}{2}\leqq\alpha$, $\beta\leqq\frac{3}{2}$ and $x_k\equiv\cos\theta_k$. Using the second asymptotic formula and the fact[4] that $n\theta_k\to j_k$ as $n\to\infty$ where j_k is the kth positive zero of $J_{\beta-1}(x)$, we find that

$$|l_k^{(n)}(1)| \to (\tfrac{1}{2}j_k)^{\beta-2}|\Gamma(\beta)J_\beta(j_k)|^{-1} \qquad \text{as } n\to\infty,\ k \text{ constant},$$

$l_1^{(n)}(-1)\to 0$ which proves the theorem:

THEOREM 7. Max $|l_1^{(n)}(x)|\to(\frac{1}{2}j_1)^{\beta-2}|\Gamma(\beta)J_\beta(j_1)|^{-1}$ *as* $n\to\infty$ (*where* $\frac{1}{2}\leqq\alpha$, $\beta\leqq\frac{3}{2}$, j_1 *is first positive zero of* $J_{\beta-1}(x)$).

A similar result holds for $l_n^{(n)}(x)$ if β is replaced by α.

For Legendre polynomials ($\alpha=\beta=1$) this limit is approximately 1.602. For $\alpha=\beta=\frac{1}{2}$ and $\alpha=\beta=\frac{3}{2}$ the limit of Theorem 7 is also an upper bound for max $|l_1^{(n)}(x)|$ and max $|l_k^{(n)}(x)|$. Whether this is true, in general, remains unanswered.

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AN INVARIANCE THEOREM FOR SUBSETS OF S^n [1]

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The purpose of this paper is to establish the following.

INVARIANCE THEOREM. *Let A and B be two homeomorphic subsets of the n-sphere S^n. If the number of components of S^n-A is finite, then it is equal to the number of components of S^n-B.*

In the case when A and B are closed this theorem is a very well known consequence of Alexander's duality theorem and its generalizations. In our case we also derive our result as a consequence of a duality theorem. However, the duality is established only for the dimension $n-1$.

Given a metric space X we shall say that Γ^k is a k-cycle in X if there is a compact subset A of X such that Γ^k is a k-dimensional convergent (Vietoris) cycle in A with coefficients modulo 2. We shall write $\Gamma^k\sim 0$ if $\Gamma^k\sim 0$ holds in some compact subset of X. The homology group of X obtained this way will be denoted by $\mathcal{H}^k(X)$; the corresponding connectivity number, by $p^k(X)$. The number $p^k(X)$ can be either finite or ∞.

[1] Presented to the Society, December 28, 1939.