

cases. In Chapter IV several theorems on the linear representations of the various rotation groups are proved by making use of the infinitesimal generators of the groups.

The first chapter of the second volume extends the spinor theory to spaces of any odd number of dimensions while the following chapter does the same for spaces of an even number of dimensions. In each of these cases the chapter closes with a brief discussion of reality conditions.

The last three chapters (totalling thirty pages) are entitled: Spinors in the Space of Special Relativity; Linear Representations of the Lorentz Group; and Spinors and Dirac Equations in Riemannian Geometry.

Comparison of this exposition of the spinor theory with the work of Veblen and his students<sup>1</sup> reveals a surprisingly small amount of common material. Thus Cartan does not employ Veblen's useful concept of a linear family of geometric transformations, which is a generalization of the well known linear family of involutions on the projective line. The avoidance of any use of the index notation is also interesting since the use of dotted indices has sometimes been regarded (quite erroneously) as a characteristic feature of the theory. These monographs do contain a full and excellent account of the group representation aspect of the theory and are an invaluable contribution to the literature of the subject.

Aside from a few misprints, the only error noted was on page 18 of the first volume, where the first lemma is false in case the fundamental quadratic form in  $n$  variables is of signature  $\pm(n-2)$ . Even in this case a weakened form of the lemma suffices for the following proofs.

WALLACE GIVENS

*Gesammelte Werke.* By Johannes Kepler. Vol. 1. *Mysterium Cosmographicum. De Stella Nova.* Edited by Max Caspar. 1938. 15+493 pp. Vol. 2. *Astronomiae Pars Optica.* Edited by F. Hammer. 1939. 467 pp. Vol. 3. *Astronomia Nova.* Edited by Max Caspar. 1937. 488 pp. Munich, Beck.

In 1936 Max Caspar published the *Bibliographia Kepleriana* as a part of this series.

The three imposing volumes of Kepler's collected works represent

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<sup>1</sup> In his introduction Cartan refers to Veblen's work as unpublished, although in his bibliography he lists Veblen's paper in the *Comptes Rendus Congrès Oslo* and the reviewer's dissertation, which was done under Veblen's direction. The most complete publication of the work at Princeton was in the form of mimeographed notes on lectures by Veblen and Givens under the title "Geometry of Complex Domains."

in a way the first conclusive steps in an established mathematically harmonious universe. The desire for simplicity in the theory of the motions of the heavenly bodies activated both Copernicus and Kepler. The former attained some simplification but retained the difficult systems of epicycles by which Greek mathematicians had achieved an explanation of celestial phenomena sufficient for some eighteen hundred years of observation. The new path in theory taken by Copernicus was the result not so much of phenomena not explainable by the old theories but of the desire for a more evident mathematical simplicity.

Man's interest in the observed movements of the heavenly bodies is doubtless the most persistent intellectual occupation of record. The ancient Babylonians derived from these observations and records material for mathematical calculations. With the Greeks comes not only the use of the older observations and records but, further, the study of associated mathematical and physical problems, connected with light and theories of vision.

In India also the heavenly bodies secured intelligent attention, with simplified methods of computation, and with algebraic devices and the use of the half-chord and even the shadow function in astronomical problems.

All of these developments contributed to the Moslem interest in astronomy, geography, and trigonometry, and the material was communicated through their works and instruments to Europe in the twelfth to fifteenth centuries. Kepler, more than any other of the European group which includes also Copernicus, Galilei, Tycho Brahe and Newton, appreciated most explicitly the achievements of the past centuries and their manifold contributions towards new theories.

Kepler began his search for a harmonious universe with an inspiration which today would be termed a delusion. However, Kepler kept faith with the observations as recorded and the continuance of this faith led him to observations not reconcilable with his inspiration nor with the theories of Ptolemy or Copernicus. The conclusive observations were furnished by Tycho Brahe, with a few by Kepler himself, and the mathematical universe involved in Kepler's laws of planetary motions gradually evolved as a result of enormous computations, treated with profound mathematical insight, establishing definitely the orbits of the planets as ellipses.

Kepler began his interest in astronomy at the University of Tübingen. The struggle for an existence led him to the casting of horoscopes and to the publication of almanacs, which were largely

for such uses. Despite this occupation the scientist Kepler sought a mathematically harmonious universe. It seemed to him to come as an inspired thought "on July 19, 1595," as he records in a letter, that there were six planets and five regular solids, and that, in consequence, the latter could be circumscribed with the sun as center about the orbits of the planets and thus harmoniously, possibly, fill in the gaps between the six planetary orbits. In the *Mysterium Cosmographicum* of 1596 Kepler elaborates this thesis, arriving at only partially successful conclusions.

In 1594 when Kepler began to teach mathematics the Copernican theory was not yet widely accepted. In his study of astronomy under an ardent Copernican, M. Mästlin, at Tübingen, Kepler had acquired an insight into the problems, leading him to desire further information concerning the size and the orbits of the planets. As the varying speeds of the planets were known and their approximate relative distances from the sun, Kepler was able to attain preliminary success in circumscribing around the recorded orbits the five regular solids, "limited by the unchangeable laws of geometry." With these five solids the sphere as a sixth solid was employed for the outermost sphere of the fixed stars.

The haste with which Kepler composed his *Mysterium Cosmographicum* is explained by the author's impending marriage and the necessity for impressing the University officials of Graz where he was employed. Doubtless had Kepler lingered, as he did with later projects, the *Mysterium* might, indeed, have never seen the light. Kepler spent some seven months in Tübingen, on vacation from Graz, completing the manuscript and arranging for the supervision of the printing by his former teacher, Mästlin.

A friendly reception was accorded the *Mysterium Cosmographicum* despite the assumption of the Copernican theory and the peculiarities inherent in the Keplerian hypothesis.

In placing the sun at the center of the universe, Kepler arrived at an appreciation, often reiterated in his later works, of the sun as the center of force of our universe. Copernicus also refers to the sun as the center of power, but he actually used in his epicycles the centers of the orbits as centers; Kepler made his measurements as far as possible directly from the sun. In Chapter XX (p. 70) Kepler refers to the sun as not only the fountain head of light, but the source of life, movement and active force (*anima*); in a later edition Kepler uses the word "*vis*."

Naturally the completed *Mysterium Cosmographicum* of 1595 was sent to the most distinguished astronomer then living, Tycho Brahe,

and the later association of Kepler with Brahe hinged upon this connection. The scholarly documentation of an untenable theory established Kepler's fame as an astronomer of great ability. For more correct planetary orbits Tycho Brahe had the data, and it was with an eye to these that Kepler made contact with the great Danish observer of the heavens.

With the *Mysterium Cosmographicum* Kepler included the *Narratio Prima* which had been published in 1540 by G. J. Rheticus,<sup>1</sup> as a preliminary statement concerning the Copernican system. To this was added an appendix by M. Mästlin, *De dimensione orbium et sphaerarum coelestium*.

The modern editors have included in this volume with the *Mysterium* four of Kepler's minor astronomical treatises, published under the title, *De Stella Nova . . .*, 1606; separate title pages are included for the other three treatises. These were publications designed in large part to take advantage, financially, of somewhat wide popular interest in astronomical occurrences.

Late in 1599 Tycho Brahe invited Kepler to visit him in Prague. In February of 1600 Kepler began somewhat tentatively employment at the Observatory in Prague.

When Kepler first approached Tycho Brahe he rather naively supposed that Tycho would be able to hand to him complete data of the orbits of the planets, so that within two or three weeks Kepler himself could check the results with his theory. In fact it took Kepler years to compute the orbit of Mars, the first one to be completed.

Tycho Brahe was having his difficulties in 1599 when he moved to Prague with the mathematical computations in connection with his observations and particularly with the orbit of Mars. Shortly thereafter Kepler was practically without a job, and as Tycho Brahe needed a mathematical astronomer, Kepler found himself in 1600 in Prague as an associate with Tycho Brahe, replacing the astronomer Longomontanus, who returned to Denmark. The observations of Mars and the problems of their reduction were placed in the hands of Kepler.

While Kepler was occupied with these computations, Tycho Brahe died October 24, 1601. Kepler was for a time prevented from free use of the observations of Tycho Brahe. Partly because of these hindrances the complete construction of the orbit of Mars was not effected until some years after the death of Tycho and then even further delayed in publication.

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<sup>1</sup> See Edward Rosen, *Three Copernican Treatises*, New York, Columbia University Press, 1939; the English translation of the Rheticus treatise is given.

In the interval Kepler attacked a number of problems in optics and in observational astronomy which were essential to any correct interpretation of the observations. Kepler's treatise on optics started with the work of Witelo as a basis. Kepler established a modern point of view for the science and even touched some of the problems shortly to be created by the introduction of the telescope.

The *Astronomiae Pars Optica* includes a wide range of discussions on the nature and properties of light, the functioning of the eye, the optical image, reflection and refraction of light rays, and astronomical optics including the lights and shadows of heavenly bodies as well as problems of parallax. Through the work of Witelo written about 1270 A.D. Kepler takes account of the work of the Arabs, notably Al-Hazan (Ibn al-Haitam) and the optics of Euclid and Ptolemy; the whole science was modernized and innumerable mistaken points of view eliminated.

Notably in connection with the observation of eclipses Kepler introduced quite new methods and hurried the book through the press in order to make his methods available to astronomers for an eclipse to occur on October 2 (12) of 1605.

Within six years after this work by Kepler appeared, observational astronomy was revolutionized by the telescope. Kepler was able to make more obvious contributions to the new theory, particularly of lenses, in his *Dioptrice* of 1611. The *Optics* of 1604 revived the problems of refraction and stimulated the definitive solution by Snell and Descartes.

The orbit of Mars had resisted the efforts of Tycho Brahe and Longomontanus. To this problem Kepler gave himself with utmost devotion over a period of five years. About Easter of 1605 Kepler abandoned the eccentric circles and the ovals to make an attempt at the ellipse as the orbit. In view of the attention given in the *Pars Optica* to the focal properties of the conics and their possible applications, one is justified in the conclusion that this study paved the way to Kepler's Laws of Planetary Motion.

While in 1601 Kepler had arrived at the orbit of Mars as in a plane inclined to the ecliptic, not until October 11 of 1605 did Kepler announce the path of Mars as an ellipse; this was done in a letter to the astronomer, Fabricius. This achievement more than any other marks the beginning of the modern theory of the movements of the heavenly bodies.

In a measure the crucial step taken by Kepler was to take the sun itself and not the center of the orbit of a planet about the sun as the center from which computations were made. In the Copernican the-

ory the center of the earth's orbit about the sun is taken as the center of the universe. Kepler's innovation moved the center (focus) to the sun itself and gave a mathematical law for the varying velocity of each planet in its orbit. Vital in the computations was new light upon the earth's orbit. In fact, the second law of Kepler that equal areas are swept out in equal times was first demonstrated by Kepler for the earth; the formulation gave the velocity as inversely proportional to the distance from the sun, the "radius-theorem." The final establishment for the orbit of Mars was dependent upon the corrections made possible by the revisions made in the earth's orbit.

In preparing for publication of Kepler's collected works translations proved to be out of the question; the editors have introduced the device of admirable summaries of contents of each volume. In addition there are notes, often giving an indication of the method of work of the author, and the genesis of the ideas; in fact, Kepler himself gives much information along this line, including even material finally rejected. The editors have also included in each volume numerous notes and for the periods concerned available lists of Kepler's correspondence, largely with brief statements of contents of the letters.

In every way the series, *Johannes Kepler Gesammelte Werke*, can be commended as worthy of the great genius of Kepler.

LOUIS C. KARPINSKI

*An Introduction to the Theory of Functions of a Real Variable.* By S. Verblunsky. Oxford, Clarendon Press, 1939. 11+169 pp. \$4.25.

This text for students and teachers was written for the special purpose of furnishing a more rigorous and accurate treatment of the elements of the theory of functions of a real variable. It is based on notes of lectures delivered by Verblunsky to students in their first year at the University of Manchester. The subject matter is entirely standard, but the treatment involves much that is new and original—ingenious and elegant proofs for certain theorems and new approaches to some parts of the subject. The dominant feature of the book is the presentation of the subject as a body of deductions from specified hypotheses.

The material treated will be sufficiently indicated by the following list of chapter headings: Chapter I, Number; II, Sets and Functions; III, Convergence; IV, Continuity and the Derivative; V, The Elementary Functions; VI, Primitives; VII, Limits and Higher Derivatives; VIII, Integrals; IX, Series. The elementary character of the book should be noted. There is no treatment of the properties of sets