ABELIAN GROUPS THAT ARE DIRECT SUMMANDS OF EVERY CONTAINING ABELIAN GROUP[1]

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It is a well known theorem that an abelian group G satisfying $G=nG$ for every positive integer n is a direct summand of every abelian group H which contains G as a subgroup. It is the object of this note to generalize this theorem to abelian groups admitting a ring of operators, and to show that the corresponding conditions are not only sufficient but are at the same time necessary. Finally we show that every abelian group admitting a ring of operators may be imbedded in a group with the above mentioned properties; and it is possible to choose this "completion" of the given group in such a way that it is isomorphic to a subgroup of every other completion.

Our investigation is concerned with abelian groups admitting a ring of operators. A ring R is an abelian group with regard to addition, its multiplication is associative, and the two operations are connected by the distributive laws. As the multiplication in R need not be commutative, we ought to distinguish left-, right- and two-sided ideals. Since, however, only left-ideals will occur in the future, we may use the term "ideals" without fear of confusion. Thus an ideal in R is a non-vacuous set M of elements in R with the property:

If m', m'' are elements in M, and if r', r'' are elements in R, then $r'm' \pm r''m''$ is an element in M.

An abelian group G whose composition is written as addition is said to admit the elements in the ring R as operators (or shorter: G is an *abelian group over* R), if with every element r in R and g in G is connected their uniquely determined product rg so that this product is an element in G and so that this multiplication satisfies the associative and distributive laws. If G is an abelian group over R, then its subgroups M are characterized by the same property as the ideals M in R.

We assume finally the existence of an element 1 in R so that $g=1g$ for every g in G and $r\cdot 1=1\cdot r=r$ for every r in R.

If x is any element in the abelian group G over R, then its *order* $N(x)$ consists of all the elements r in R which satisfy $rx=0$. One verifies that every order $N(x)$ is an ideal in R, and that $N(x)=R$ if, and only if, $x=0$.

If M is an ideal in R, and if x is an element in G, then a subgroup

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of G is formed by the elements mx for m in M; and this subgroup may be denoted by Mx. (It is a subgroup of the cyclic group generated by x.) The correspondence between the element m in M and the element mx in Mx is a special case of the homomorphisms of M into G. Here a *homomorphism* ϕ of the ideal M in R into the abelian group G over R is defined as a single-valued function m^ϕ of the elements m in M with values in G which satisfies

$$(r'm' \pm r''m'')^\phi = r'(m'^\phi) \pm r''(m''^\phi)$$

for m', m'' in M and r', r'' in R.

We are now ready to state and prove our main result.[2]

THEOREM 1. *The following two properties of an abelian group G over the ring R are each a consequence of the other.*

(a) *If G is a subgroup of the abelian group H over R, then G is a direct summand of H.*

(b) *To every ideal M in R and to every homomorphism ϕ of M into G there exists some element v in G so that $m^\phi = mv$ for every m in M.*

PROOF. Assume first that (a) is satisfied by G. If M is an ideal in R, and if ϕ is a homomorphism of M into G, then there exists one and essentially only one group H over R which is generated in adjoining to G an element h, subject to the relations

$$mh = m^\phi \qquad \text{for every } m \text{ in } M.$$

It is a consequence of (a) that H is the direct sum of G and of a suitable subgroup K of H so that every element in H may be represented in one and only one way in the form: $g+k$ for g in G and k in K. This applies in particular to the element h so that $h=v+w$ for

[2] The following is a remark by the referee: "It is perhaps of some interest to observe that Theorem 1 contains a generalization of the theorem that every representation of a semisimple algebra is fully reducible. Indeed, how does one characterize those rings R such that *every* abelian group G admitting R as an operator ring has the property (a)? The answer is that every left-ideal in R must be generated by an idempotent element, and this is equivalent to saying that R is semisimple (both chain conditions and no radical).

"The sufficiency of this condition is proved by showing that G has the property (b). Let $m \rightarrow m^\phi$ be any homomorphism of a left-ideal M of R into G. Since $M=Re$ with $e^2=e$, then $me=m$ for every m in M, and if we take $v=e^\phi$ we have $m^\phi=(me)^\phi=me^\phi=mv$.

"The necessity is proved by observing that every left-ideal M of R is itself an abelian group over R, and is a subgroup of R. Hence to each M there exists a complementary left-ideal N. From $1=e+e'$ with unique e in M, e' in N, one concludes in the usual way that $e^2=e$ and $M=Re$."

uniquely determined elements v and w in G and K respectively. Then $m^\phi = mh = mv + mw$ is for every element m in M an element in G, and so is mv and $m^\phi - mv = mw$. Since mw is an element in K, it follows that $mw = 0$ or $m^\phi = mv$ for every m in M; and this shows that (b) is a consequence of (a).

Assume now conversely that (b) is satisfied by the abelian group G over R, and that G is a subgroup of the abelian group H over R. Then there exists a greatest subgroup K of H whose meet with G is 0. The subgroup S of H which is generated by G and K is their direct sum; and hence it suffices to prove that $H = S = G + K$.

If $S \neq H$, then there exists in H an element w that is not contained in S. The coset $W = S + w$ is an element in the quotient-group H/S and its order $N(W)$ is an ideal in R. If m is any element in $N(W)$, then mw is an element in S; and it follows from the construction of S that there exist uniquely determined elements $g(m)$ and $k(m)$ in G and K respectively so that

$$mw = g(m) + k(m) \qquad \text{for } m \text{ in } N(W).$$

Thus a homomorphism of $N(W)$ into G is defined in mapping the element m in $N(W)$ upon the element $g(m)$ in G. There exists therefore by condition (b) an element v in G so that $mv = g(m)$ for every m in $N(W)$. The element $w' = w - v$ consequently satisfies

$$S + w' = S + w = W$$

and

$$mw' = k(m)$$

for every element m in $N(W)$. Since w is not an element in S, neither is w'. Since K is a greatest subgroup of H whose meet with G is 0, and since w' is not an element in S and therefore not in K, adjoining w' to K generates a subgroup whose meet with G is different from 0. Hence there exists an element k in K, and an element r in R so that

$$k + rw' = g \neq 0$$

is an element in G. Since $rw' = g - k$ is an element in S, it follows that r is an element in $N(W)$ so that $g = k + k(r)$ is an element in the meet of G and K. This contradicts, however, the construction of K and the choice of g. Our hypothesis $S \neq H$ has thus led us to a contradiction; and this completes the proof.

If M is an ideal in the ring R, and if G is an abelian group over R, then G is termed *M-complete*, if there exists to every homomorphism ϕ of M into G an element v in G so that $m^\phi = mv$ for every m in M. In this terminology, condition (b) of Theorem 1 states that G is M-complete for every ideal M in R. It is our object to characterize the M-complete groups, provided M is a principal ideal. For this end we need several notations.

If M is an ideal in R, then denote by G_M the set of all the elements g in G which satisfy $mg=0$ for every m in M. Note that G_M need not be a subgroup of the abelian group G over R, though it is closed with regard to addition and subtraction.

If p is any element in R, then pG consists of all the elements pg for g in G. Note again that pG need not be a subgroup of the abelian group G over R.

The principal ideal in R, generated by the element p in R, consists of all the elements rp for r in R and may therefore be denoted by Rp; and $N(p)$ consists of all the elements r in R so that $rp=0$. $N(p)$ is clearly an ideal in R.

THEOREM 2. *The abelian group G over R is Rp-complete if and only if* $G_{N(p)} \leqq pG$.

PROOF. Suppose first that G is Rp-complete. If x is an element in $G_{N(p)}$, then $r'p=r''p$ implies $r'x=r''x$, since the first equation is equivalent to the fact that $r'-r''$ is an element in $N(p)$. Thus a homomorphism of Rp into G is defined in mapping rp upon rx. Since G is Rp-complete, there exists an element v in G so that $rpv=rx$ for every r in R. This implies in particular that $pv=x$, that is, our condition is necessary.

Suppose conversely that our condition be satisfied by G. If ϕ is a homomorphism of Rp into G, then p^ϕ is an element in $G_{N(p)}$, since we have $r(p^\phi)=(rp)^\phi=0$ for elements r in $N(p)$. Hence there exists an element q in G so that $p^\phi=pq$; and clearly $(rp)^\phi=r(p^\phi)=r(pq)=(rp)q$ for every r in R so that G is Rp-complete.

COROLLARY 1. *If p is an element in R so that $N(p)=0$, then $G=pG$ is a necessary and sufficient condition for Rp-completeness of the abelian group G over R.*

This is a consequence of Theorem 2, since $G_0=G$.

COROLLARY 2. *If $N(p)=0$ for every element $p\neq 0$ in the ring R, and if every ideal in R is a principal ideal Rp, then $G=pG$ for every $p\neq 0$ in R is a necessary and sufficient condition for the abelian group G over R to be a direct summand of every abelian group H over R which contains G as a subgroup.*

This is an obvious consequence of Theorem 1 and Corollary 1.

If, in particular, R consists of the rational integers, then the hypotheses of Corollary 2 are satisfied. In this case the sufficiency of the condition of the Corollary 2 has been known for a long time.[3]

An abelian group G over the ring R is termed *complete*, if it is M-complete for every ideal M in R. Thus the complete groups are just the groups satisfying the properties (a) and (b) of Theorem 1.

THEOREM 3. *Every abelian group over the ring R is a subgroup of a complete abelian group over the ring R.*

PROOF. If G is an abelian group over the ring R, M an ideal in R, and ϕ a homomorphism of M into G, then there exists an abelian group H over R which contains G as a subgroup and which contains an element x so that $mx = m^{\phi}$ for every m in M.

By repetition of the construction of the preceding paragraph one may show that if G is an abelian group over the ring R, then there exists an abelian group G' over the ring R which contains G as a subgroup and which satisfies the following condition:

(3.1) If M is an ideal in R, and if ϕ is a homomorphism of M into G, then there exists an element v in G' so that $mv = m^{\phi}$ for every m in M.

Denote now by λ an ordinal number which is a limit-ordinal and whose cardinal number is greater than the number of elements in R. Then it follows from the second paragraph of the proof that there exists for every ordinal ν with $0 \leqq \nu \leqq \lambda$ an abelian group G_ν over R with the following properties:

(i) $G_0 = G$;

(ii) $G_\nu \leqq G_\mu$ for $\nu < \mu$;

(iii) G_ν is for limit-ordinals ν the set of all the elements contained in groups G_μ for $\mu < \nu$;

(iv) G_ν and $G_{\nu+1}$ satisfy condition (3.1).

Suppose now that M is an ideal in R and that ϕ is a homomorphism of M into $H = G_\lambda$. Then there exists an ordinal $\sigma < \lambda$ so that G_σ contains all the elements m^{ϕ}; and there exists therefore an element v in $G_{\sigma+1}$ so that $mv = m^{\phi}$ for every m. H is therefore complete.

THEOREM 4. *To every subgroup G of the complete abelian group K over the ring R there exists a complete subgroup G^* of K which contains G as a subgroup and which satisfies the following condition:*

(E) *Every isomorphism of G upon a subgroup of a complete abelian group H over R is induced by an isomorphism of G^* upon a subgroup of H.*

[3] For a comparable proof see Annals of Mathematics, (2), vol. 37 (1936), pp. 766–767, (1; 1).

PROOF. If T is an abelian group over the ring R, if M is an ideal in R, then the homomorphism ϕ of M into T is termed *reducible* in T, if there exists an ideal M' in R and a homomorphism ϕ' of M' into T so that $M < M'$ and so that ϕ and ϕ' coincide on M. If ϕ is not reducible, then it is irreducible in T.

(4.1) *The abelian group T over the ring R is complete, if there exists to every ideal M in R and to every irreducible homomorphism ϕ of M into T an element v in T so that $mv = m^{\phi}$ for every m in M.*

To prove this statement let J be an ideal in R and γ a homomorphism of J into T. Then there exists a greatest ideal M in R so that $J \leqq M$ and so that γ is induced in J by a homomorphism ϕ of M into T. It is clear that ϕ is irreducible in T. Hence there exists an element v in T so that $mv = m^{\phi}$ for every m in M. This implies however that $nv = n^{\phi} = n^{\gamma}$ for every n in J, that is, T is complete.

It is a consequence of (4.1) and of the completeness of K that there exists an ascending chain of subgroups G_{ν} for $0 \leqq \nu \leqq \lambda$ with the following properties:

(i) $G = G_0$;

(ii) $G_{\nu} \leqq K$ for $\nu \leqq \lambda$;

(iii) $G_{\nu+1}$ is generated by adjoining to G_{ν} an element g_{ν} with the following properties:

(iii′) The homomorphism of $N(G_{\nu}+g_{\nu})$ into G_{ν} which is defined by mapping the element m in $N(G_{\nu}+g_{\nu})$ upon the element mg_{ν} in G_{ν} is irreducible in G_{ν}.

(iii″)[4] G_{ν} does not contain any element x so that $mx = mg_{\nu}$ for every m in $N(G_{\nu}+g_{\nu})$.

(iv) G_{ν} is for limit-ordinals ν the set of all the elements contained in groups G_{μ} for $\mu < \nu$.

(v) $G_{\lambda} = G^*$ is complete.

We are now going to prove that this subgroup G^* of K satisfies condition (E). Thus assume that ρ is an isomorphism of G upon the subgroup $G' = G^{\rho}$ of the complete group H. We are going to construct subgroups G'_{ν} of H and isomorphisms ρ_{ν} of G_{ν} upon G'_{ν} with the following properties:

(1) $G' = G'_0$, $\rho = \rho_0$;

(2) $G'_{\nu} \leqq G'_{\mu}$ for $\nu \leqq \mu$;

(3) ρ_{ν} and ρ_{μ} coincide on G_{ν} for $\nu \leqq \mu$.

In order to prove the possibility of this construction it suffices to show the existence of $G'_{\nu+1}$, $\rho_{\nu+1}$ under the hypothesis of the existence of G'_{ν}, ρ_{ν}.

[4] This condition (iii″) is not really needed for the proof, though it is convenient for the construction of the chain G_{ν}.

A homomorphism irreducible in G_ν' of $M=N(G_\nu+g_\nu)$ into G_ν' is defined by mapping the element m in M upon the element $m^\phi=(mg_\nu)^{\rho_\nu}$. Since H is complete, there exists an element h in H so that $m^\phi=mh$ for every m in M. If $M'=N(G_\nu'+h)$, then it is clear that $M\leqq M'$. If m is in M', then mh is an element in G_ν'. Thus a homomorphism γ of M' into G_ν is defined by mapping the element m in M' upon the element $m^\gamma=(mh)^{\rho_\nu^{-1}}$. If, in particular, m is an element in M, then $m^\gamma=mg_\nu$; and it follows from (iii′) that $M=M'$. Suppose now that g' is an element in G_ν' and u an element in R so that $g'+uh=0$. Then u is an element in $M=M'$ and it follows from the above considerations that $-g'=uh=u^{\gamma\rho_\nu}=(ug_\nu)^{\rho_\nu}$. Hence there exists one and only one isomorphism $\rho_{\nu+1}$ of $G_{\nu+1}$ upon the group $G_{\nu+1}'$, generated by G_ν' and h, which isomorphism induces ρ_ν in G_ν and maps g_ν upon h.

Thus there exists finally an isomorphism ρ_λ of $G^*=G_\lambda$ upon G_λ' which induces ρ in G; and this completes the proof.

COROLLARY. *Assume that K is a smallest complete abelian group over the ring R containing the subgroups G_i. Then G_1 and G_2 are isomorphic if, and only if, there exists an automorphism of K mapping G_1 upon G_2.*

This is an obvious consequence of Theorem 4. It should be noted, however, that the complete group G^*, satisfying (E) and containing G, whose existence is assured by Theorems 3 and 4, is only "essentially smallest," but need not be "actually smallest."

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