NEUTRAL ELEMENTS IN GENERAL LATTICES[1]

GARRETT BIRKHOFF

1. **Introduction.** O. Ore has defined "neutral" elements in modular lattices as elements a satisfying $a \cap (x \cup y) = (a \cap x) \cup (a \cap y)$ for all x, y and dually.[2] In the case of complemented modular lattices, the neutral elements compose the "center" in J. von Neumann's theories of continuous geometries and regular rings—that is, the set of elements having unique complements.[3]

The purpose of the present note is to extend the notion of neutral elements to general lattices. More precisely, call an element a of a lattice "neutral" if and only if every triple $\{a, x, y\}$ generates a distributive sublattice. It is proved that the neutral elements of any lattice L form a distributive sublattice, consisting of the elements carried into $[I, O]$ under isomorphisms of L with sublattices of direct products. Actually, this sublattice is the intersection of the maximal distributive sublattices of L.

Further, complements of neutral elements, when they exist, are unique and neutral. The sublattice of complemented neutral elements may be called the "center" of a lattice: it consists of those elements carried into $[I, O]$ under isomorphisms of L with direct products.

2. **Fundamental definition.** We define an element a of a lattice L to be "neutral" if and only if every triple $\{a, x, y\}$ generates a distributive sublattice of L.

LEMMA 1. *If a is "neutral," then the dual correspondences $x \to x \cap a$ and $x \to x \cup a$ are endomorphisms[4] of L.*

PROOF. By definition, $(x \cup y) \cap a = (x \cup a) \cap (y \cup a)$ and $(x \cap y) \cap a = (x \cap a) \cap (y \cap a)$, and dually. We note that this condition, which is sufficient to guarantee neutrality in the case of modular lattices, does not guarantee neutrality in general—see the graph below.

---

[1] Presented to the Society, September 8, 1939.

[2] O. Ore, *On the foundations of abstract algebra* I, Annals of Mathematics, (2), vol. 36 (1935), pp. 406–437. For the definitions of lattices and modular lattices (called by Ore structures and Dedekind structures), as well as of sublattice, distributive lattice, O, I, and so on, compare the author's *Lattices and their applications*, this Bulletin, vol. 44 (1938), pp. 793–800—or the author's *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, 1940.

[3] J. von Neumann, *Lectures on Continuous Geometries*, Princeton, 1935–1936. Cf. also R. P. Dilworth, *Note on complemented modular lattices*, this Bulletin, vol. 45 (1939), pp. 74–76.

[4] We define an endomorphism as a homomorphism of L with itself.

LEMMA 2. *If a is neutral, then $x \cap a = y \cap a$ and $x \cup a = y \cup a$ imply $x = y$.*

PROOF. By direct computation, using the distributive law twice,

$$x = x \cap (x \cup a) = x \cap (y \cup a) = (x \cap y) \cup (x \cap a)$$
$$= (x \cap y) \cup (y \cap a) = y \cap (x \cup a) = y \cap (y \cup a) = y.$$

Using x in the graph, we see that this condition by itself is also not sufficient. However, Lemmas 1–2 together are sufficient to guarantee neutrality.

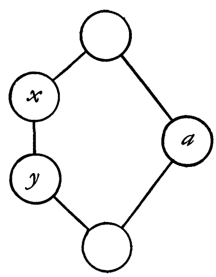

FIG. 1

Indeed, consider the correspondence $x \rightarrow [x \cap a, x \cup a]$ from L to the product[5] ST of the sublattice S of elements $s \leqq a$, with the sublattice T of elements $t \geqq a$. By Lemma 1, it is homomorphic onto a sublattice of ST; by Lemma 2, it is one-one; hence it is isomorphic. Moreover $x \rightarrow [a, a] = [I, O]$, since a is the I of S and the O of T.

But conversely, $[I, O]$ is obviously "neutral" in ST, since each component is. Hence it is neutral in every sublattice of ST, including L, and we conclude[6] that the following holds:

THEOREM 1. *An element of a lattice is neutral if and only if it is carried into $[I, O]$ under an isomorphism of the lattice with a sublattice of a direct product.*

3. **Neutral elements a sublattice.** Just as in the case of modular lattices, we have the following theorem:

THEOREM 2. *The neutral elements of any lattice form a distributive sublattice.*

PROOF. Let a and b be neutral. Then since the product of two

[5] By the "product" ST, is meant the system of couples $[s, t]$, $s \in S$, $t \in T$, where $[s, t] \cap [s', t'] = [s \cap s', t \cap t']$ and dually.

[6] N.B., we do *not* assume that L itself has an O or an I.

endomorphisms of L is an endomorphism, it follows that the correspondence

$$x \rightarrow [x \cap a \cap b, (x \cup a) \cap b, (x \cap a) \cup b, x \cup a \cup b]$$

defines a homomorphism of L onto a sublattice of a product $STUV$ of four sublattices of itself. Moreover by Lemma 2, $(x \cap a)$ and $(x \cup a)$ and therefore x, are determined uniquely by their images under the homomorphism. Hence the homomorphism is an isomorphism. But $a \cup b$ goes into $[I, I, I, O]$ under this—and hence into $[I, O]$ of $(STU)V$. Thus $a \cup b$ is neutral; dually, $a \cap b$ is neutral, which was to be proved.

4. **Intersection of maximal distributive sublattices.** The set of neutral elements of a lattice is also characterized in another way by the next theorem:

THEOREM 3. *The set of neutral elements of a lattice L is the intersection of its maximal distributive sublattices.*

PROOF. First, if a is not neutral, then some triple $\{a, x, y\}$ is not distributive. Hence *no* maximal distributive sublattice obtained by enlarging the distributive sublattice generated by $\{x, y\}$ can contain a. Consequently, the intersection of the maximal distributive sublattices of L contains no non-neutral elements.

Conversely, if a is neutral, and S is a distributive sublattice of L, consider the sublattice generated by $\{a, S\}$. The endomorphisms $x \rightarrow x \cap a$ and $x \rightarrow x \cup a$ carry it into sublattices generated by a distributive sublattice and I or O. But such sublattices are always distributive—hence so is the sublattice generated by $\{a, S\}$, since it is a sublattice of a product of distributive lattices. Thus every *maximal* distributive sublattice contains a, and the intersection of the maximal distributive sublattices contains every neutral element (as well as no non-neutral elements).

5. **Center of a lattice.** When one comes to complements of neutral elements, one finds that the following statement holds:

THEOREM 4. *Complements of neutral elements, when they exist, are unique and neutral.*

PROOF. Using Theorem 1, we see that $[I, O] \cap [x, y]$ is $[O, O]$ if and only if $x = O$, while $[I, O] \cup [x, y]$ is $[I, I]$ if and only if $y = I$. Hence $[I, O]$ has no complement except $[O, I]$ in the sublattice of ST isomorphic with L, proving uniqueness. Moreover $[O, I]$ is itself neutral, completing the proof.

COROLLARY 1. *The neutral elements of a complemented lattice form a Boolean algebra.*

We define the "center" of a lattice as the set of its complemented neutral elements.

THEOREM 5. *The center of any lattice L is a complemented distributive sublattice—and hence a Boolean algebra.*

PROOF. If a and b are neutral elements of L, with (neutral) complements a' and b', then

$$(a \cap b) \cap (a' \cup b') = (a \cap b \cap a') \cup (a \cap b \cap b') = O \cup O = O,$$
$$(a \cap b) \cup (a' \cup b') = (a \cup a' \cup b') \cap (b \cup a' \cup b') = I \cap I = I,$$

and so $a \cap b$ is complemented. Dually, $a \cup b$ is complemented, completing the proof.

We can now specialize Theorem 1 by proving the following:

THEOREM 6. *An element is in the center of a lattice L if and only if it is carried into* $[I, O]$ *under an isomorphism of L with a direct product.*

PROOF. By Theorem 1, such an element is neutral, and it has the complement $[O, I]$. Conversely, suppose a and a' are complementary neutral elements of L. Then for all x,

$$x = x \cap I = x \cap (a \cup a') = (x \cap a) \cup (x \cap a').$$

Hence the correspondence $x \rightarrow [x \cap a, x \cap a']$ is one-one between L and the couples $[u, v]$ with $u \leqq a$, $v \leqq a'$; the inverse correspondence is $[u, v] \rightarrow u \cup v$. But it obviously preserves inclusion; hence it is an *isomorphism*. Finally, it carries L into the product ST of the lattice S of elements $s \leqq a$, with the lattice T of elements $t \leqq a'$, while it carries a into $[a \cap a, a \cap a'] = [I, O]$ in ST.

HARVARD UNIVERSITY