sults in an important field, which will inspire and stimulate further research.

J. SHOHAT

Structure of Algebras. By A. Adrian Albert. (American Mathematical Society Colloquium Publications, vol. 24.) New York, American Mathematical Society, 1939. 11+210 pp.

The study of algebras has been one of the most significant features of the present period in the history of mathematics; and in the theory of algebras a monument has been erected recording some of the characteristic traits of contemporary mathematical thought.

One may say that algebras have been pushed into the centre of attention by the publication of Dickson's *Algebras and their Arithmetics*; and from that moment on they have kept the interest of the mathematical public. In the meantime a great number of the problems has been solved, methods have been streamlined so that a moment propitious for the survey of the results has arrived. One of the principal actors in the movement has given an account of its results. The mathematical public certainly will be grateful for his effort, as he has been able to capture the inherent beauty of the theory of algebras and to communicate it to the reader.

The book may be divided roughly into two parts, the first being concerned with the general theory, the second containing applications to related problems. It should be mentioned at once that the theory of representations has been "put in its place"; that is, it appears as an application of the general theory and has not been used for the derivation of the results of the general theory.

The general theory of algebras may be defined as that part of the theory in which no special restrictions are imposed upon the field of reference. There are two main topics of discussion. The first is the reduction to simple algebras and the second the discussion of the simple algebras themselves.

The reduction theory proceeds in two steps. One shows first the existence of a radical and the semisimplicity of the algebra modulo its radical. This semisimple difference algebra may be represented by some subalgebra of the original algebra, provided the difference algebra stays semisimple under every scalar extension (this result belongs to a later phase of the theory, since changes of the field of reference have to be considered). But since both the investigation of nilpotent algebras and of the possible extensions of a nilpotent algebra by a semisimple algebra have started only recently, the state of the theory allows one only to speak of a theory of semisimple algebras and not yet of a theory of algebras. The second step in the reduction theory is the proof of the theorems that every semisimple algebra possesses an identity element and is the direct sum of uniquely determined (not only in the sense of equivalence) simple algebras; and thus it is clear how to construct the most general semisimple algebra, as soon as the simple algebras are known. It should be noted that the theorems concerning semisimple algebras are gotten as special cases of corresponding theorems concerning algebras which may or may not possess a radical. However the reduction contained in these more general theorems does not lead to a class of algebras as tractable as the simple ones.

Every simple algebra is a direct product of a total matric algebra and a division algebra; and any two representations of this kind are conjugate, that is, may be transformed into each other by means of inner automorphisms. The inner automorphisms of simple algebras are exactly those automorphisms which leave the elements of the centrum invariant. The centrum of a simple algebra being a field, it may be taken as the field of reference, in which case we say that the the algebra is normal simple. Then the above theorem states that all the automorphisms of a normal simple algebra are inner. This theorem is supplemented by the fact that any equivalence of two simple subalgebras of a normal simple algebra may be induced by an automorphism of the containing algebra, provided the three algebras have the same identity element.

It follows from the above that the structure of a normal simple algebra is completely determined by two invariants: the one numerical, namely, the degree of the total matric algebra in question, the other a structural invariant, namely, the structure of the division algebra mentioned above. Thus one may be led to the following construction. One considers all the normal simple algebras over a fixed field F. Those normal simple algebras over F which are only distinguished by the nature of the total matric algebra involved and lead to the same division algebra are put into one class and these classes form an abelian group under direct multiplication (of its elements).

If the field N is a finite, normal, and separable extension of the field F, then N determines a subgroup of the group of classes of normal simple algebras over F as follows. There exist some normal simple algebras over F which contain N as a maximal subfield. The product of such an algebra by N is a total matric algebra over N, that is, is split by N; and each such algebra is a crossed product of N by the

Galois group of N over F. The classes of normal simple algebras over F which are split by N form a subgroup of the group of classes of algebras over F; and this subgroup is practically the same as the group of classes of associated factor sets of N over F. As each normal simple algebra over F is split by some such N, it suffices to consider these subgroups of the group of classes; and thus it is possible to use the theory of factor sets for proving theorems like the following one: every element of the group of classes is of finite order in this group; this order divides the degree of the characteristic division algebra and is divisible by every prime divisor of this degree.

As to the methods of proof employed in the derivation of this theory two tools ought to be mentioned which the author uses with much dexterity and success. There is first the commutator of a subalgebra in a containing algebra, that is, the set of all those elements in the containing algebra which permute with every element in the given subalgebra—the adoption of this term commutator may be regretted, as this concept is called centralizer everywhere outside this theory. The importance of the commutator may be seen from the following theorems. If some algebra is the direct product of two algebras B and C such that B is normal simple and such that B and C have the same identity, then C is the commutator of B in their direct product, that is, one factor of the direct product determines the other one in a unique fashion. More generally: the simple subalgebras of a normal simple algebra occur in pairs such that each member of such a pair is the commutator of the other one; and forming the commutator is an involutorial operation as far as simple subalgebras of a normal simple algebra are concerned.

Observing that multiplication of the elements of an algebra by a fixed element is a linear transformation of the algebra, that the linear transformations form a total matric algebra, that the transformations gotten by left-multiplication with an element form exactly the commutator of those gotten by right-multiplication (and conversely), and that the second of these sets of transformations is isomorphic, the first anti-isomorphic to the given algebra, one proves the theorem that the direct product of a normal simple algebra by its reciprocal (anti-isomorphic) is a total matric algebra. This theorem is so powerful a weapon in the theory because it permits a reduction of the proofs of theorems concerning normal simple algebras to the proofs of the corresponding theorems concerning total matric algebras.

The following partial converse of the last theorem ought to be mentioned. The direct product of two division algebras is a total matric algebra if, and only if, these two division algebras are reciprocal.

As an illustration of the author's methods in handling these tools we indicate a proof of the theorem that every automorphism of a normal simple algebra is an inner automorphism (our reason for choosing just this example is that the author does not use his methods for a proof of this theorem and that the proof given in the text is inconclusive). If A is a normal simple algebra, A' its reciprocal, then their direct product is a total matric algebra $T=A\times A'$. If f is an automorphism of A, then there exists one and only one automorphism of T which induces f in A and leaves every element in A' invariant. As every automorphism of a total matric algebra is an inner automorphism, there exists some element t in T such that transformation by t induces in T just this automorphism. This element belongs clearly to the commutator of A' in T. Since the commutator of A' in T is just A, it follows that t belongs to A so that f is the inner automorphism of A, induced by this element t.

As a first application of the theory of algebras, the theory of representations has been treated. This theory is no longer concerned with the study of the structure of algebras but with the study of the structure of homomorphisms between algebras. If both algebras are considered over the same field, then there exists however a connection between the two theories which is very near to an equivalence. The following theorems may be mentioned as typical examples of this situation. A representation of the algebra A over F in the subalgebra A^* of a total matric algebra M over F is irreducible if, and only if, A^* is simple and the commutator of A^* in M is a division algebra; it is fully decomposable if, and only if, A^* is semisimple; it stays irreducible under every scalar extension if, and only if, A^* itself is a total matric algebra. On the other hand it must be admitted that by its very definition the theory of representations is much more complicated than the theory of algebras proper; and thus both theories are greatly improved by doing the simpler thing first and using it then for the derivation of the more involved theory.

The most important application of the general theory and perhaps the most interesting part of the whole work is the enumeration of the normal simple algebras over finite algebraic number fields or—what amounts to the same thing—of the simple algebras over the field of rational numbers. The problem of how to construct all these algebras is settled by the famous theorem that every simple algebra over the field of rational numbers is not only similar to, but may actually be represented as, the crossed product of a cyclic extension of an algebraic number field by its cyclic Galois group. In order to classify these algebras the author proceeds as follows. If the field F is a finite extension of the field of rational numbers, V a valuation of F and A a normal simple algebra over F, then denote by F_V the essentially uniquely determined derived field of F with respect to the valuation V and by A_V the direct product of A and F_V (that is, the scalar extension of A by F_V). A reduction of the problem "in the large" to a problem "in the small" is contained in the theorem that the normal simple algebras A and B over F are equivalent if, and only if, A_V and B_V are equivalent for every valuation V. This theorem is a comparatively simple consequence of the following lemma (which the author states without giving a proof): an element in F is the norm of an element in the given finite and normal extension N of prime degree over F if, and only if, it is a norm of an element in N_V for every valuation V of N. Thus it suffices to determine these algebras A_V. There exists at most a finite number of valuations V of F such that A_V is not a total matric algebra. If A_V is not a total matric algebra, and if V is an archimedian valuation, then A_V is a real quaternion algebra. If A_V is not a total matric algebra, and if V is a non-archimedian valuation, then A_V is completely determined by two numerical invariants one of which is just the degree of the division algebra characteristic for the class of A_V whereas the other one is derived from a certain normal form of representing A_V as a crossed product.

There is a great number of further equally interesting applications of the theory which find treatment in this book. But it would lead too far to discuss them here at great length. Suffice it to mention such topics as the cyclic systems, the modern theory of Riemann matrices and of involutions, and more special problems like the enumeration of all normal division algebras of degrees three and four.

The standard source of reference for the theory of algebras has been in recent years Deuring's report on this topic. As to the material covered none of these works has been a subset of the other and even in their treatment of the common parts they differ widely. In the reviewer's opinion Albert's text is the more easily accessible one of the two, so much so that he feels that the book will serve admirably as an introduction into the theory to all those who know the rudiments of present-day algebra. The expert on the other hand will find this exposition interesting, stimulating and useful both because of the new material included and because of the more "streamlined" treatment of it.

An extensive bibliography of recent publications on related topics has been added which will be welcome and helpful to every student of the field.

REINHOLD BAER