BOOK REVIEWS

Orthogonal Polynomials. By Gabor Szegö. (American Mathematical Society Colloquium Publications, vol. 23.) New York, American Mathematical Society, 1939. 10+401 pp.

The general concept of orthogonal polynomials (OP) was introduced in Analysis in the fifties of the 19th century by Tchebyscheff, then professor at the University of St. Petersbourg. Naturally his pupils have carried on his research in this field. Thus in 1868 J. Sokhotsky in his thesis gives a general approach to OP, in some respects even more general than the one which prevails today. In 1884 Markoff's thesis again treats the general OP corresponding to absolutely continuous as well as discrete distribution-functions. In 1886 C. Possé published a monograph: *Sur quelques applications des fractions continues algébriques* where he considers the general OP related to

$$\int_a^b [f(y)/(x-y)]dy.$$

None of these works may be considered as a systematic treatment of the subject. In fact, time was not yet ripe for such treatment (it took almost a hundred years after the introduction of Legendre polynomials before a systematic treatment of these appeared—Heine's *Handbuch der Kugelfunktionen*). In 1932 N. Abramesco published an article in the Annales de Toulouse on the general OP. In 1934 the writer of these lines published a monograph in the *Mémorial des Sciences Mathématiques* designed to treat the theory of OP (applications are reserved for a second monograph in this collection) systematically and with as many details as limitation of space permitted.

The book under review gives in 400 pages a detailed and systematic treatment of the theory and applications of OP. It offers a very lucid and elegant exposition of the subject, to which Szegö himself made so many contributions. It is by no means a compilation of results already known. It presents much material which is new and important; many old results are presented in a novel setting: more precise or more general statements, new proofs. The author tried to bring the book as much up-to-date as possible and has generally succeeded.

Szegö's book consists of 16 chapters, the content of which may be summarized as follows. In Chapter I the author is marshaling the necessary analytical tools: representation of nonnegative polynomials, bounds for the derivative of a polynomial or trigonometric sum, trigonometric and polynomial approximation of continuous functions, the concept of closure, linear functional operators, the Gamma-function, Bessel's functions. We also find here interesting "comparison theorems" of Sturm's type for linear differential equations. Chapter II treats the existence of OP and their representation in determinantal and multiple-integral form. The starting point here is the orthogonality property. The general principles are illustrated on the classical OP—of Jacobi, Hermite and Laguerre—also on OP corresponding to some special distribution-functions absolutely continuous and discrete. Chapter III deals with the general properties of OP: some extremal properties, closure, integral approximation in the sense of the least squares and p-powers, the all-important recurrence-relation, Darboux' formula. We also find here a discussion of the elementary properties of the zeros of OP and the corresponding mechanical quadrature formula. For the coefficients of the latter formula the author derives, in three different ways, the celebrated Tchebyscheff inequalities ("separation theorem"). The chapter closes with a brief discussion of algebraic continued fractions, in their relation to OP. Chapters IV, V, VIII, IX are devoted to the classical OP. In Chapters IV–V we find their differential equation, also representation by means of trigonometric and Bessel functions, hypergeometrical series, contour-integrals. In Chapter VIII the author makes use of the powerful tools of modern analysis in order to obtain asymptotic expressions for the classical OP in various parts of the interval of orthogonality (with an estimate of the remainder), also in the complex domain. This yields asymptotic estimates of their zeros. The interesting feature is the use of Bessel's functions which offers the advantage of extending the above asymptotic formulas to the end-points of the orthogonality interval—of great importance in the study of expansion problems. Such expansions in series of the classical OP are taken up in Chapter IX: ordinary convergence and (C, k) summability for various values of k. These two chapters form, we believe, one of the most interesting parts of the book. Many results are new; many known results are presented in a decidedly more precise form. The study is based upon the corresponding Lebesgue constants. A masterly use of these constants enables the author to derive conditions for convergence and summability which, from the point of view of generality and precision, leave practically nothing to be desired. This is in sharp contrast to many older investigations, where the said conditions are frequently of an artificially restricted character. The author is the more justified in devoting so much attention to the classical OP, for they form the backbone of the whole theory and serve as a yardstick in dealing with the general OP.

We turn back to Chapter VI. Here a study is made of the zeros of OP: analytic distribution, variation with a parameter. The general results are considerably elaborated for the classical OP (discriminant, extremal properties), where use is made of the corresponding differential equation (Sturm's methods). The results are extended to Fejér's generalization of Legendre polynomials, also to the generalized (non-orthogonal) Jacobi polynomials. In Chapter VII the author derives various inequalities for some special classes of OP (classical OP, monotonic weight-function) and related functions. These inequalities are of importance in mechanical quadratures, interpolation, also in evaluating certain integrals involving OP. We find here also a refinement, for polynomials, of the mean-value theorem of the integral calculus. Chapters X–XIII and XVI, based in the main upon the modern theory of functions of a complex variable, contains results due almost entirely to Szegö. Chapter X is devoted to the analytical representation of positive functions. This is made use of in the next chapter for a study of OP on the unit-circle (these were introduced in Analysis by Szegö). This discussion is continued and generalized in Chapters XII and XIII. It is extended in Chapter XVI to a rectifiable Jordan curve. The simple transformation $x=\frac{1}{2}(z+z^{-1})$ enables the author to return to the real domain and to derive important results concerning OP on a finite segment of the real axis. We note the asymptotic expression for a certain class of OP on $(-1, 1)$ corresponding to the so-called trigonometric weight-function (S. Bernstein). Szegö's method of deriving the said asymptotic expression is much simpler than that of Bernstein. Several general equiconvergence theorems for the real and complex domains constitute another important contribution. Chapters XIV and XV deal with interpolation and closely related mechanical quadrature (they are less extensive than other parts of the book). Here the author makes extensive use of the results developed in the preceding chapters. In Chapter XV a study is made of Lagrange and Hermite interpolation formulae based upon the zeros of OP: convergence, mean-convergence in the sense of pth powers. For the classical OP Szegö derives results more precise than those obtained by other writers. In Chapter XVI the author studies, mainly for the classical OP, the convergence of mechanical quadrature formulae based on Lagrange interpolation. Two types of mechanical quadrature formulae are considered: with and without weight-function. Concerning the latter type, the author takes up the question of positiveness of the coefficients ("Cotes' numbers") which is of cardinal importance in this theory.

In closing the author gives a stimulating set of problems and exercises, also a considerable bibliography.

It goes without saying that the writer of a scientific book is the supreme judge in choosing its content. It is only natural for him to favor his own brain-children and, from amongst the works of others, those closely related to his own.

In the Introduction the author disclaims completeness of treatment. This disarming remark is necessary but not sufficient. In a book of this kind one would like to find as complete a treatment as possible, preparing the reader for future progress in various branches of the subject.

The omission of the problem of moments is, we believe, very regrettable. The problem of moments is important historically and is growing in importance both in pure and applied analysis. Its inclusion would have been much to the point. Moreover, it would have enabled the author (on the basis of the important results of M. Riesz) to give a complete treatment of closure for an infinite interval (the importance of closure in the theory of OP cannot be underestimated). As it is, the author discusses closure, in case of an infinite interval, for Laguerre and Hermite polynomials only. Furthermore, this inclusion of the moment-problem would have made possible a discussion of the limiting values of certain important definite integrals. Another regrettable feature is, we believe, the far too short space ($3\frac{1}{2}$ pages) devoted to continued fractions. The author remarks that continued fractions have been gradually abandoned as a starting point for the theory of OP. Be that as it may, one should not underrate continued fractions as a powerful tool in the theory and application of OP. We may mention a few other omissions: OP in several variables; other methods of summability of series of OP in addition to (C, k) summability, for example, the interesting results of Hille concerning Abel summability of expansions in series of Laguerre polynomials; application of OP to algebra (zeros of polynomials).

The reviewer did not pursue the thankless task of hunting misprints, minor errors or minor faults in exposition. While no book is free of these, the known thoroughness of the author offers sufficient assurance in this respect.

With all these omissions the beautifully printed book of Szegö is an excellent addition to the Colloquium Publications of the Society. It is a remarkable source of powerful methods and far-reaching results in an important field, which will inspire and stimulate further research.

J. SHOHAT

Structure of Algebras. By A. Adrian Albert. (American Mathematical Society Colloquium Publications, vol. 24.) New York, American Mathematical Society, 1939. 11+210 pp.

The study of algebras has been one of the most significant features of the present period in the history of mathematics; and in the theory of algebras a monument has been erected recording some of the characteristic traits of contemporary mathematical thought.

One may say that algebras have been pushed into the centre of attention by the publication of Dickson's *Algebras and their Arithmetics*; and from that moment on they have kept the interest of the mathematical public. In the meantime a great number of the problems has been solved, methods have been streamlined so that a moment propitious for the survey of the results has arrived. One of the principal actors in the movement has given an account of its results. The mathematical public certainly will be grateful for his effort, as he has been able to capture the inherent beauty of the theory of algebras and to communicate it to the reader.

The book may be divided roughly into two parts, the first being concerned with the general theory, the second containing applications to related problems. It should be mentioned at once that the theory of representations has been "put in its place"; that is, it appears as an application of the general theory and has not been used for the derivation of the results of the general theory.

The general theory of algebras may be defined as that part of the theory in which no special restrictions are imposed upon the field of reference. There are two main topics of discussion. The first is the reduction to simple algebras and the second the discussion of the simple algebras themselves.

The reduction theory proceeds in two steps. One shows first the existence of a radical and the semisimplicity of the algebra modulo its radical. This semisimple difference algebra may be represented by some subalgebra of the original algebra, provided the difference algebra stays semisimple under every scalar extension (this result belongs to a later phase of the theory, since changes of the field of reference have to be considered). But since both the investigation of nilpotent algebras and of the possible extensions of a nilpotent alge-