To separate real and imaginary parts we may write

$$(f_1' + if_1)(f_2' + if_2) = f_1'f_2' - f_1f_2 + i\frac{d}{dx}(f_1f_2).$$

Differentiating this last expression p times and taking absolute magnitudes, we obtain

$$\left\{\frac{d^{p+1}}{dx^{p+1}}(f_1f_2)\right\}^2 + \left\{\frac{d^p}{dx^p}(f_1'f_2' - f_1f_2)\right\}^2 \leqq (2^p)^2,$$

which is more than we set out to prove. The functions $f_1(x) = \sin(x+\alpha_1)$ and $f_2(x) = \sin(x+\alpha_2)$ show that our constant is the "best possible."

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ON THE CARATHÉODORY CONDITION FOR UNILATERAL VARIATIONS[1]

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The two formulations and proofs of the Carathéodory condition in the calculus of variations given by Graves[2] do not necessarily apply to the case when the minimizing curve may have arcs in common with the boundary of the region of admissible variations.[3] The purpose of this note is to show how his first formulation and proof can be modified so as to be applicable to unilateral (one-sided) variations in the plane.

An admissible curve

$$E_0:\quad x^\alpha = x^\alpha(t), \qquad t_0 \leqq t \leqq T,\ \alpha = 1, 2, \cdots, n,$$

which minimizes the integral

$$J = \int_{t_0}^{T} F(x_1, \cdots, x_n, x_1', \cdots, x_n')dt \equiv \int_{t_0}^{T} F(x, x')dt$$

in the class of all admissible curves joining two fixed points x_0 and X in space of n dimensions $(n>1)$, must satisfy certain well known con-

[1] Presented to the Society, December 29, 1939.

[2] *Discontinuous solutions in space problems of the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), pp. 13–19.

[3] Cf. Mancill, *The minimum of a definite integral with respect to unilateral variations*, Contributions to the Calculus of Variations, 1933–37, University of Chicago, 1937, p. 121, condition C_{3C}.

ditions.[4] Suppose that E_0 is an extremaloid joining the points x_0 and X and having corners at $x_1, \cdots, x_m$, and suppose that the function F_1 is different from zero along E_0 including both sides of corners. If the function

$$\Omega_0(x, p^-, p^+) = (\partial F^+/\partial x^\alpha)p^{\alpha-} - (\partial F^-/\partial x^\alpha)p^{\alpha+},$$

where $p^{\alpha-}$ and $p^{\alpha+}$ represent the direction cosines of the tangents to E_0 at the corners, is different from zero at each corner of E_0, there exists an $(n-1)$-parameter family

$$x^\alpha = \phi^\alpha(t, a) \tag{1}$$

of extremaloids defined for $t_0 - \delta \leqq t \leqq T + \delta$, passing through the point x_0 for $t = t_0$, and containing[5] E_0 for $a^i = a_0^i$, $t_0 \leqq t \leqq T$. We shall now prove the following form of the Carathéodory condition:

THEOREM. *Let E_0 be an extremaloid joining the points x_0 and X and minimizing the integral J, satisfying the following conditions:*

(1) *E_0 is positively strong;*[6]

(2) *$\Omega_0 \neq 0$ at the corners on E_0;*

(3) *E_0 is uniquely determined by each of its elements (x, x');*[7]

(4) *the determinant*

$$D(t, a_0) \equiv \left| \frac{\partial \phi^\alpha}{\partial a^i} \quad \frac{\partial \phi^\alpha}{\partial t} \right|_{a=a_0}$$

does not vanish at the corners on E_0.

Then the determinant $D(t, a_0)$ does not change sign at the corners on E_0.

Suppose that the determinant $D(t, a_0)$ changes sign at a corner x_k where $t = t_k$. Also, suppose that the corners on all extremaloids of the family occur for fixed values of the parameter[8] t. Consider the one-parameter family of curves E_u constructed by Graves.[9] The value of the integral J taken along E_u is a function $J(u)$ whose derivative near $u = t_k$ is

$$J'(u) = -\mathcal{E}(\phi^{\alpha-}, \phi'^{\alpha-}, x'^{\alpha+}) \equiv -\mathcal{E}[\bar{t}(u), \bar{a}(u), x'^{\alpha+}(u)]. \tag{2}$$

The second derivative of $J(u)$ *at* $u = t_k$ is

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[4] For a full statement of the problem and the classical necessary conditions, see Graves, loc. cit., pp. 2–3.

[5] Graves, loc. cit., p. 6.

[6] Cf. Graves, loc. cit., p. 9.

[7] Ibid., p. 11.

[8] Ibid., p. 7.

[9] Ibid., p. 14.

(3) $$J''(t_k) = - [(\partial F^+/\partial x^\alpha)x'^{\alpha-} - (\partial F^-/\partial x^\alpha)x'^{\alpha+}]\bar{t}'(t_k),$$

on account of the extremal property of E_0, and since the derivative of the function $\mathcal{E}[t_k, \bar{a}(u), x'^{\alpha+}(u)]$ is zero at $u=t_k$. This last statement follows from assumptions (1) and (3) of the theorem and the fact that we have assumed that the corners on all the extremaloids of the family occur for fixed values of the parameter t, and therefore the function $\mathcal{E}[t_k, \bar{a}(u), x'^{\alpha+}(u)] \geqq 0$ near $u=t_k$ and equals zero at t_k. By making use of the homogeneity property of the function F and the expression for $\bar{t}'(t_k)$ given by Graves,[10] the derivative (3) may be reduced to

(4) $$J''(t_k) = - [(x'^{\alpha-}x'^{\alpha-})(x'^{\alpha+}x'^{\alpha+})]^{1/2} \cdot \Omega_0(x, p^-, p^+)D(t_k + 0, a_0)/D(t_k - 0, a_0) < 0,$$

where $p^{\alpha-} = x'^{\alpha-}/(x'^{\alpha-}x'^{\alpha-})^{1/2}$ and $p^{\alpha+} = x'^{\alpha+}/(x'^{\alpha+}x'^{\alpha+})^{1/2}$. Consequently, $J(t_k)$ *is a maximum of the function* $J(u)$ *for* u *in a neighborhood of* t_k, since $J'(t_k)=0$ and $J''(t_k)<0$. Thus, E_0 could not minimize the integral J.

Let us now consider the problem of unilateral variations in the plane. In this case it can be shown that a Carathéodory condition as stated here applies to every arc of the minimizing curve *which is an arc of an extremaloid*. The condition may be stated in terms of the one-parameter family of extremaloids through any fixed point of such an arc. For such arcs of the minimizing curve as E_{23} immediately following a non-extremal arc E_{12} of the boundary of the region of admissible curves whose direction at the point 2 where it meets E_{12} is not directed towards the exterior side of E_{12} but is as shown in Fig. 1, the Carathéodory condition may be stated in terms of the one-parameter family of extremaloids whose members, at least those members on the admissible side of the minimizing curve, are tangent to the arc E_{12} of the boundary.[11] In order to insure the existence of the families of extremaloids just described, it is necessary to assume that the continuity and homogeneity properties of the integrand F hold in an extended region containing in its interior those arcs of the minimizing curve which are extremaloids in common with the boundary. It should also be pointed out that in the construction of the family of extremaloids tangent to E_{12}, it is necessary to assume that the Weierstrass $\mathcal{E}$-function is greater than or equal to zero along

[10] Loc. cit., p. 15.

[11] Mancill, loc. cit., pp. 105–107. For a discussion of the case when the extremaloid is directed towards the exterior side of E_{12} see pages 138–141. In this case the minimizing curve need not satisfy the corner conditions at the point 2.

E_{12} preceding 2.[12] The family of extremaloids is then composed of the family of extremals tangent to E_{12} in a neighborhood of and preceding the point 2, and the family of extremaloids containing the extrema-

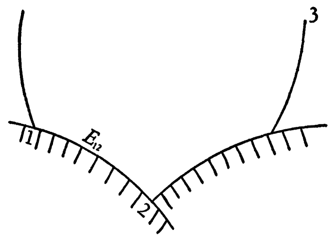

FIG. 1

loid E_{23} which is the *continuation* family of the extremals tangent to E_{12} near the point 2.

In order to show how the proof of the theorem applies in the way just described, let (1) represent the one-parameter family of extremaloids through a fixed point of such an arc of the boundary. Let x_k be the *first corner beyond this fixed point.* Then the family of curves E_u used in the proof of the theorem exists for u near t_k and its members are admissible curves for u on one side of t_k. Therefore, the relation (2) holds here for u on the admissible side of t_k. The function $-\mathcal{E}[\bar{t}(u), \bar{a}(u), x'^{\alpha+}(u)]$ is defined locally and in a neighborhood of t_k. Its derivative at $u=t_k$ is $J''(t_k)$ as given in (4). Consequently, it follows from (2) that $J(t_k)$ is greater than $J(u)$ for all values of u on the admissible side of t_k and sufficiently near t_k. The same argument can now be made at each succeeding corner in turn.

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[12] Mancill, ibid., p. 107. For a proof of the Weierstrass condition along E_{12}, see Graves, American Mathematical Monthly, vol. 41 (1934), pp. 502–504.