SOME PROBLEMS IN INTERPOLATION BY CHARACTERISTIC FUNCTIONS OF LINEAR DIFFERENTIAL SYSTEMS OF THE FOURTH ORDER

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In this paper we consider the convergence to $f(x)$, defined on $[0, 1]$, of

$$\Sigma_p[f(x)] = \alpha_{0p}u_0(x) + \alpha_{1p}u_1(x) + \cdots + \alpha_{pp}u_p(x),$$

where $u_n(x)$, $(n=0, 1, \cdots, p)$, are characteristic functions of certain self-adjoint linear differential systems of fourth order,

$$\alpha_{np} = \sum_{k=0}^{p}{}' f(x_k)u_n(x_k)\left\{ \sum_{k=0}^{p}{}' u_n^2(x_k)\right\}^{-1}, \qquad n = 0, 1, \cdots, p,$$

and the symbol $\sum'$ is used in the sense $\sum_{k=0}^{\prime p} y_k = y_0/2 + \sum_{k=1}^{p} y_k$. Throughout the discussion, $x_k = 2k/(2p+1)$, $(k=0, 1, \cdots, p)$. The differential systems considered are

$$u^{(\mathrm{iv})} - \rho^4 u = 0,$$

with boundary conditions

I. $u'(0)=0$, $u'''(0)=0$, $u'(1)=0$, $u'''(1)+u(1)=0$,
II. $u'(0)=0$, $u'''(0)=0$, $u'(1)+u(1)=0$, $u'''(1)+u''(1)=0$,
III. $u(0)=0$, $u''(0)=0$, $u(1)=0$, $u''(1)+u'(1)=0$,
IV. $u'(0)=0$, $u'''(0)=0$, $u(1)=0$, $u''(1)+u'(1)=0$,
V. $u(0)=0$, $u'(0)=0$, $u(1)=0$, $u'(1)=0$,
VI. $u'(0)=0$, $u'''(0)=0$, $u(1)=0$, $u'(1)=0$.

The following theorems may be proved for these systems respectively.

I, II. *If $f(x)$ is continuous and of bounded variation in $[0, 1]$, then* $\lim_{p\to\infty}\Sigma_p[f(x)]=f(x)$ *uniformly in* $[0, 1]$.

III. *If $f(x)$ is continuous and of bounded variation in $[0, 1]$ and* $f(0)=f(1)=0$, *then* $\lim_{p\to\infty}\Sigma_p[f(x)]=f(x)$ *uniformly in* $[0, 1]$.

IV. *If $f(x)$ is continuous and of bounded variation in $[0, 1]$ and* $f(1)=0$, *then* $\lim_{p\to\infty}\Sigma_p[f(x)]=f(x)$ *uniformly in* $[0, 1-\eta]$.

V, VI. *If $f(x)$ satisfies a Lipschitz condition in $[0, 1]$ and* $f(0)=f(1)=0$, *then* $\lim_{p\to\infty}\Sigma_p[f(x)]=f(x)$ *uniformly in* $[\eta, 1-\eta]$.

Here and hereafter $\eta>0$ is arbitrarily small but fixed.

The method of proof for these theorems, as well as for those to follow, is essentially the same as that of C. M. Jensen [1], who considered convergence properties of a somewhat similar sum using Sturm-Liouville functions.

The following equivalence theorems (the term "equivalence" is used in the sense of Jensen) may be proved for systems I, II, III, IV, respectively.

I, II. *If $f(x)$ is continuous in* $[0, 1]$, *then*

$$\lim_{p\to\infty} \{\Sigma_p[f(x)] - T_p[f(x)]\} = 0$$

uniformly in $[0, 1]$, *where* $T_p[f(x)]$ *is the cosine interpolation formula*

$$T_p[f(x)] = a_{0p} + a_{1p} \cos \pi x + \cdots + a_{pp} \cos p\pi x,$$

$$a_{np} = \sum_{k=0}^{p}{}' f(x_k) \cos n\pi x_k \left\{ \sum_{k=0}^{p}{}' \cos^2 n\pi x_k \right\}^{-1}$$

$$= \begin{cases} \dfrac{4}{2p+1} \displaystyle\sum_{k=0}^{p}{}' f(x_k) \cos n\pi x_k, & n = 1, 2, \cdots, p, \\ \dfrac{2}{2p+1} \displaystyle\sum_{k=0}^{p}{}' f(x_k), & n = 0. \end{cases}$$

III. *If $f(x)$ is continuous in* $[0, 1]$, *then*

$$\lim_{p\to\infty} \{\Sigma_p[f(x)] - \overline{T}_p[f(x)]\} = 0$$

uniformly in $[\eta, 1-\eta]$, *where* $\overline{T}_p[f(x)]$ *is the sine interpolation formula*

$$\overline{T}_p[f(x)] = \bar{a}_{1p} \sin \pi x + \bar{a}_{2p} \sin 2\pi x + \cdots + \bar{a}_{pp} \sin p\pi x,$$

$$\bar{a}_{np} = \sum_{k=0}^{p}{}' f(x_k) \sin n\pi x_k \left\{ \sum_{k=0}^{p}{}' \sin^2 n\pi x_k \right\}^{-1}$$

$$= \frac{4}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \sin n\pi x_k, \qquad n = 1, 2, \cdots, p.$$

IV. *If $f(x)$ is continuous in* $[0, 1]$, *then*

$$\lim_{p\to\infty} \{\Sigma_p[f(x)] - U_p[f(x)]\} = 0$$

uniformly in $[\eta, 1-\eta]$, *where* $U_p[f(x)]$ *is given by*

$$U_p[f(x)] = b_{0p} \cos (\pi/2)x + b_{1p} \cos (3\pi/2)x + \cdots + b_{pp} \cos (p + 1/2)\pi x,$$

$$b_{np} = \sum_{k=0}^{p}{}' f(x_k) \cos (n + 1/2)\pi x_k \left\{ \sum_{k=0}^{p}{}' \cos^2 (n + 1/2)\pi x_k \right\}^{-1}$$

$$= \begin{cases} \dfrac{4}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \cos (n + 1/2)\pi x_k, & n = 0, 1, \cdots, p-1, \\ \dfrac{2}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \cos (n + 1/2)\pi x_k, & n = p. \end{cases}$$

The first set of theorems also holds using characteristic functions of $u^{(\mathrm{iv})} - [\rho^4 + \lambda(x)]u = 0$, with the above boundary conditions, $\lambda(x)$ being an arbitrary continuous function. Here, provided ${}_0u(x) \not\equiv 0$, the term ${}_0u(x) \sum_{k=0}^{\prime p} f(x_k)\, {}_0u(x_k) \{\sum_{k=0}^{\prime p} {}_0u^2(x_k)\}^{-1}$ is adjoined to $\Sigma_p[f(x)]$, where ${}_0u(x)$ is the characteristic function corresponding to $\rho = 0$. (For $\lambda(x) \equiv 0$, $\rho = 0$ is not a characteristic number.)

We give here the proof of the theorem for system V. System V is the problem of lateral vibrations of an elastic homogeneous rod clamped at both ends [**2**]. We first state two lemmas.

LEMMA 1. *If $f(x)$ is continuous and of bounded variation in* $[0, 1]$ *and $f(0) = f(1) = 0$, then $W_p[f(x)]$ and $w_p[f(x)]$ tend to the same limit $M(x)$ uniformly in* $[\eta, 1-\eta]$ *as $p \to \infty$, where*

$$\begin{aligned} W_p[f(x)] = {} & c_{0p}[\cos (\pi/2)x - \sin (\pi/2)x] \\ & + c_{1p}[\cos (3\pi/2)x - \sin (3\pi/2)x] + \cdots \\ & + c_{pp}[\cos (p + 1/2)\pi x - \sin (p + 1/2)\pi x], \end{aligned}$$

$$c_{np} = \frac{2}{2p+1} \sum_{k=0}^{p}{}' f(x_k) [\cos (n + 1/2)\pi x_k - \sin (n + 1/2)\pi x_k], \qquad n = 0, 1, \cdots, p,$$

$$\begin{aligned} w_p[f(x)] = {} & c_0[\cos (\pi/2)x - \sin (\pi/2)x] + c_1[\cos (3\pi/2)x - \sin (3\pi/2)x] \\ & + \cdots + c_p[\cos (p + 1/2)\pi x - \sin (p + 1/2)\pi x], \end{aligned}$$

$$c_n = \int_0^1 f(t) [\cos (n + 1/2)\pi t - \sin (n + 1/2)\pi t] dt, \qquad n = 0, 1, \cdots, p.$$

LEMMA 2. *If $f(x)$ satisfies a Lipschitz condition in* $[0, 1]$ *and $f(0) = f(1) = 0$, then there exists a constant C, depending only on η, such that for n sufficiently great, $p \geqq n$, and x in* $[\eta, 1-\eta]$,

$$| \alpha_{np} u_n(x) - c_{np}[\cos (n + 1/2)\pi x - \sin (n + 1/2)\pi x] | < C/n^2.$$

To prove Lemma 1, write

$$W_p[f(x)] = {}_1W_p[f(x)] + {}_2W_p[f(x)] - {}_3W_p[f(x)],$$
$$w_p[f(x)] = {}_1w_p[f(x)] + {}_2w_p[f(x)] - {}_3w_p[f(x)],$$

where

$$ {}_1W_p[f(x)] = \frac{2}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \sum_{n=0}^{p} \cos (n + 1/2)\pi x_k \cos (n + 1/2)\pi x, $$

$$ {}_2W_p[f(x)] = \frac{2}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \sum_{n=0}^{p} \sin (n + 1/2)\pi x_k \sin (n + 1/2)\pi x, $$

$$ {}_3W_p[f(x)] = \frac{2}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \sum_{n=0}^{p} \sin (n + 1/2)\pi(x_k + x), $$

$$ {}_1w_p[f(x)] = \int_0^1 f(t) \sum_{n=0}^{p} \cos (n + 1/2)\pi t \cos (n + 1/2)\pi x dt, $$

$$ {}_2w_p[f(x)] = \int_0^1 f(t) \sum_{n=0}^{p} \sin (n + 1/2)\pi t \sin (n + 1/2)\pi x dt, $$

$$ {}_3w_p[f(x)] = \int_0^1 f(t) \sum_{n=0}^{p} \sin (n + 1/2)\pi(t + x) dt. $$

We first show that if $f(x)$ is continuous and of bounded variation in $[0, 1]$, then $\lim_{p\to\infty} {}_1W_p[f(x)] = (1/2)f(x)$ uniformly in $[0, 1-\eta]$. We employ the cosine interpolation formula $T_p[f(x)]$. We shall use $r_n(x)$ as generic notation for a function uniformly bounded in n and for x in $[0, 1]$, unless the range for x is otherwise stated, and r_n $[r_{np}]$ for a quantity depending on n $[n$ and $p]$ and uniformly bounded in n $[n$ and $p]$. We have

$$ \begin{aligned} T_p[f(x)] &= \frac{2}{2p+1} \sum_{k=0}^{p}{}' f(x_k) + \frac{4}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \sum_{n=1}^{p} \cos n\pi x_k \cos n\pi x \\ &= \frac{1}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \left[\frac{\sin (p + 1/2)\pi(x_k - x)}{\sin (\pi/2)(x_k - x)} \right. \\ &\qquad\qquad \left. + \frac{\sin (p + 1/2)\pi(x_k + x)}{\sin (\pi/2)(x_k + x)} \right] \\ &= \frac{1}{2p+1} \sum_{k=0}^{p}{}' f(x_k) \left[\frac{\sin p\pi(x_k - x) \cos (\pi/2)(x_k - x)}{\sin (\pi/2)(x_k - x)} \right. \\ &\qquad\qquad \left. + \frac{\sin p\pi(x_k + x) \cos (\pi/2)(x_k + x)}{\sin (\pi/2)(x_k + x)} \right] \\ &\quad + \frac{2}{2p+1} \cos p\pi x \sum_{k=0}^{p}{}' f(x_k) \cos p\pi x_k. \end{aligned} $$

Using auxiliary Lemma A stated below, we have

$$T_p[f(x)] = \frac{1}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\left[\frac{\sin p\pi(x_k - x)\cos(\pi/2)(x_k - x)}{\sin(\pi/2)(x_k - x)} + \frac{\sin p\pi(x_k + x)\cos(\pi/2)(x_k + x)}{\sin(\pi/2)(x_k + x)}\right] + \frac{r_p(x)}{p},$$

and

$$\begin{aligned} {}_1W_p[f(x)] &= \frac{2}{2p+1}\sum_{k=0}^{p}{}' f(x_k)\sum_{n=0}^{p-1}\cos(n+1/2)\pi x_k \cos(n+1/2)\pi x \\ &\quad + r_p(x)/p \\ &= \frac{1}{2(2p+1)}\sum_{k=0}^{p}{}' f(x_k)\left[\frac{\sin p\pi(x_k - x)}{\sin(\pi/2)(x_k - x)} + \frac{\sin p\pi(x_k + x)}{\sin(\pi/2)(x_k + x)}\right] + \frac{r_p(x)}{p}. \end{aligned}$$

Thus

$$\begin{aligned} &{}_1W_p[f(x)] - (1/2)T_p[f(x)] \\ &= \frac{1}{2(2p+1)}\sum_{k=0}^{p}{}' f(x_k)\left[\frac{1-\cos(\pi/2)(x_k - x)}{\sin(\pi/2)(x_k - x)}\sin p\pi(x_k - x) + \frac{1-\cos(\pi/2)(x_k + x)}{\sin(\pi/2)(x_k + x)}\sin p\pi(x_k + x)\right] + \frac{r_p(x)}{p} \\ &= \frac{1}{2(2p+1)}\sum_{k=0}^{p}{}' f(x_k)\left[\tan(\pi/4)(x_k - x)\sin p\pi(x_k - x) + \tan(\pi/4)(x_k + x)\sin p\pi(x_k + x)\right] + r_p(x)/p. \end{aligned}$$

By reason of the nature of Lemma A and the fact that for t in $[0, 1]$ the functions $\tan(\pi/4)(t-x)$ and $\tan(\pi/4)(t+x)$ are of uniform bounded variation with respect to x in $[0, 1-\eta]$, we have ${}_1W_p[f(x)] - (1/2)T_p[f(x)] = r_p(x)/p$ for x in $[0, 1-\eta]$. Hence [**3**] $\lim_{p\to\infty} {}_1W_p[f(x)] = (1/2)f(x)$ uniformly in $[0, 1-\eta]$.

Employing $t_p[f(x)]$, the partial sum of order p in the Fourier cosine series,

$$t_p[f(x)] = a_0 + a_1\cos\pi x + \cdots + a_p\cos p\pi x,$$

$$a_n = \frac{\int_0^1 f(x)\cos n\pi x dx}{\int_0^1 \cos^2 n\pi x dx} = \begin{cases} 2\int_0^1 f(x)\cos n\pi x dx, & n = 1, 2, \cdots, p, \\ \int_0^1 f(x)dx, & n = 0, \end{cases}$$

we may show similarly that if $f(x)$ is continuous and of bounded variation in $[0, 1]$, then $\lim_{p\to\infty} {}_1w_p[f(x)] = (1/2)f(x)$ uniformly in $[0, 1-\eta]$. Employing $\overline{T}_p[f(x)]$, the sine interpolation formula, and $\bar{t}_p[f(x)]$, the partial sum of order p in the Fourier sine series,

$$\bar{t}_p[f(x)] = \bar{a}_1 \sin \pi x + \bar{a}_2 \sin 2\pi x + \cdots + \bar{a}_p \sin p\pi x,$$

$$\bar{a}_n = \frac{\int_0^1 f(x) \sin n\pi x dx}{\int_0^1 \sin^2 n\pi x dx} = 2\int_0^1 f(x) \sin n\pi x dx, \quad n = 1, 2, \cdots, p,$$

we may likewise show that if $f(x)$ is continuous and of bounded variation in $[0, 1]$ and $f(0) = f(1) = 0$, then

$$\lim_{p\to\infty} {}_2W_p[f(x)] = (1/2)f(x), \qquad \lim_{p\to\infty} {}_2w_p[f(x)] = (1/2)f(x)$$

uniformly in $[0, 1-\eta]$. In connection with ${}_1w_p[f(x)]$ and ${}_2w_p[f(x)]$ we use Lemma B, stated below.

Finally, we show that if $f(x)$ is of bounded variation in $[0, 1]$, then ${}_3W_p[f(x)]$ and ${}_3w_p[f(x)]$ both tend to $(1/2)\int_0^1 f(t)\{\sin (\pi/2)(t+x)\}^{-1}dt$ uniformly in $[\eta, 1-\eta]$. We have

$${}_3w_p[f(x)] = \frac{1}{2}\int_0^1 f(t) \frac{1 - \cos (p+1)\pi(t+x)}{\sin (\pi/2)(t+x)} dt.$$

For x in $[\eta, 1-\eta]$,

$$\begin{aligned} {}_3w_p[f(x)] = &\frac{1}{2}\int_0^1 \frac{f(t)}{\sin (\pi/2)(t+x)} dt \\ &- \frac{1}{2}\int_0^1 f(t) \frac{\cos (p+1)\pi(t+x)}{\sin (\pi/2)(t+x)} dt. \end{aligned}$$

Effecting some trigonometric reductions on the second integrand and using Lemma B, we have

$${}_3w_p[f(x)] = \frac{1}{2}\int_0^1 \frac{f(t)}{\sin (\pi/2)(t+x)} dt + \frac{r_p(x)}{p}.$$

We may then prove that

$$\lim_{p\to\infty} {}_3W_p[f(x)] = \frac{1}{2}\int_0^1 \frac{f(t)}{\sin (\pi/2)(t+x)} dt$$

uniformly in $[\eta, 1-\eta]$, using the fact just established that

$$\lim_{p\to\infty} {}_3w_p[f(x)] = \frac{1}{2}\int_0^1 \frac{f(t)}{\sin (\pi/2)(t+x)}\,dt$$

uniformly in $[\eta, 1-\eta]$, together with Lemma C, stated below.

LEMMA A. *If $f(x)$ is of bounded variation in $[0, 1]$, then for $n=1, 2, \cdots, 2p$,*

$$\frac{1}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\cos n\pi x_k\right| = \frac{r_{np}}{n},$$

$$\frac{1}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\sin n\pi x_k\right| = \frac{r_{np}}{n}\cdot$$

Also

$$\frac{1}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\sin (2p+1)\pi x_k\right| = 0.$$

LEMMA B. *If $f(x)$ is of bounded variation in $[0, 1]$, then for $n>0$,*

$$\left|\int_0^1 f(x)\cos n\pi x dx\right| = \frac{r_n}{n}, \qquad \left|\int_0^1 f(x)\sin n\pi x dx\right| = \frac{r_n}{n}\cdot$$

LEMMA C. *If $f(x)$ is of bounded variation in $[0, 1]$, then for any preassigned $\epsilon>0$ there exists Q such that for $p>q\geqq Q$ and x in $[\eta, 1]$,*

$$\frac{2}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\sum_{n=q+1}^{p}\sin (n+1/2)\pi(x_k+x)\right| < \epsilon.$$

Lemma 2 is proved by means of auxiliary Lemmas D, E, and A.

LEMMA D. *For n sufficiently great, and $p\geqq n$,*

$$\left[\sum_{k=0}^{p}{}' u_n^2(x_k)\right]^{-1} = \frac{2}{2p+1}\left(1+\frac{r_{np}}{n}\right).$$

LEMMA E. *If $f(x)$ satisfies a Lipschitz condition in $[0, 1]$ and if $f(0)=f(1)=0$, then for $n=1, 2, \cdots, p$,*

$$\frac{1}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\exp\left\{-(n+1/2)\pi x_k\right\}\right| = \frac{r_{np}}{n^2},$$

$$\frac{1}{2p+1}\left|\sum_{k=0}^{p}{}' f(x_k)\exp\left\{-(n+1/2)\pi(1-x_k)\right\}\right| = \frac{r_{np}}{n^2}\cdot$$

The following asymptotic expression is known [4] for the characteristic functions of system V:

$$\begin{aligned} u_n(x) = {} & \cos (n + 1/2)\pi x - \sin (n + 1/2)\pi x \\ & + (-1)^n \exp \{-(n + 1/2)\pi(1 - x)\} \\ & - \exp \{-(n + 1/2)\pi x\} + \exp \{-(n + 1/2)\pi\} r_n(x). \end{aligned}$$

Define

$$\sigma_p[f(x)] = \alpha_0 u_0(x) + \alpha_1 u_1(x) + \cdots + \alpha_p u_p(x),$$

$$\alpha_n = \frac{\int_0^1 f(x) u_n(x) dx}{\int_0^1 u_n^2(x) dx}, \qquad n = 0, 1, \cdots, p.$$

Denote by $\Sigma_p^{(r,s)}$ the sum of the terms in $\Sigma_p[f(x)]$ with subscripts r to s inclusive; similarly in the other sums.

For x in $[\eta, 1-\eta]$,

$$\begin{aligned} |f(x) - \Sigma_p[f(x)]| \leqq {} & |\sigma^{(0,N)} - \Sigma_p^{(0,N)}| + |W_p^{(N+1,p)} - \Sigma_p^{(N+1,p)}| \\ & + |W_p^{(0,N)} - w^{(0,N)}| + |M(x) - W_p^{(0,p)}| \\ & + |w^{(0,N)} - M(x)| + |f(x) - \sigma^{(0,N)}|. \end{aligned}$$

The right-hand member can be made arbitrarily small by choosing p sufficiently large. Call the six terms $D_1, D_2, \cdots, D_6$. Given ϵ, first choose N sufficiently large so that $D_2 < \epsilon$, $D_5 < \epsilon$, $D_6 < \epsilon$, for $p \geqq N+1$. Having fixed N, choose P sufficiently large so that $D_1 < \epsilon$, $D_3 < \epsilon$, $D_4 < \epsilon$ for $p \geqq P$. It remains to justify these statements. For D_6 we use a result in a paper by J. D. Tamarkin [5]. For D_2 we use Lemma 2, and for D_4 and D_5 Lemma 1. In D_1 and D_3 we are dealing essentially with integrals of continuous functions and the sums which tend to the integrals as limits, the number of terms being finite.

Now consider $u^{(\text{iv})} - [\rho^4 + \lambda(x)]u = 0$, $\lambda(x)$ being an arbitrary continuous function. A fundamental system of solutions of $u^{(\text{iv})} - \rho^4 u = 0$ is $u_1 = \cos \rho x$, $u_2 = \sin \rho x$, $u_3 = e^{\rho x}$, $u_4 = e^{-\rho x}$. By the method of variation of constants we have, as an equation satisfied by u,

$$\begin{aligned} u = {} & A \cos \rho x + B \sin \rho x + Ce^{\rho x} + De^{-\rho x} \\ & + \frac{1}{2\rho^3} \int_0^x \lambda(t) u(t) \left[-\sin \rho(x - t) + \frac{e^{\rho(x-t)}}{2} - \frac{e^{-\rho(x-t)}}{2} \right] dt, \end{aligned}$$

where A, B, C, D are arbitrary constants. Applying the boundary conditions of system V and choosing the multiplicative constant so that $A=1$, we have, assuming $\rho\neq 0$,

$$
\begin{aligned}
u = {} & \cos \rho x - \sin \rho x \\
& + \Bigg[\sin \rho x \Big\{ - 2e^{-\rho} \sin \rho + \frac{1}{2\rho^3}\int_0^1 \lambda(t)u(t)\,[- e^{-\rho} \sin \rho(1 - t) \\
& \qquad\qquad - e^{-\rho} \cos \rho(1 - t) + e^{-\rho t}]dt \Big\} \\
& + e^{-\rho(1-x)} \sin \rho - \frac{1}{4\rho^3}\int_0^1 \lambda(t)u(t)\,[- e^{-\rho(1-x)} \sin \rho(1 - t) \\
& \qquad\qquad - e^{-\rho(1-x)} \cos \rho(1 - t) + e^{\rho(x-t)}]dt \\
& - e^{-\rho x} + e^{-\rho(1+x)} \cos \rho + \frac{1}{4\rho^3}\int_0^1 \lambda(t)u(t)\,[- e^{-\rho(1+x)} \sin \rho(1 - t) \\
& - e^{-\rho(1+x)} \cos \rho(1 - t) + e^{-\rho(x+t)}]dt \Bigg] \big\{1 - e^{-\rho} \sin \rho - e^{-\rho} \cos \rho\big\}^{-1} \\
& + \frac{1}{2\rho^3}\int_0^x \lambda(t)u(t)\left[- \sin \rho(x - t) + \frac{e^{\rho(x-t)}}{2} - \frac{e^{-\rho(x-t)}}{2}\right] dt,
\end{aligned}
$$

and, as an equation satisfied by the characteristic numbers $\rho=\rho_n$,

$$
\begin{aligned}
\cos \rho = {} & 2e^{-\rho} - e^{-2\rho} \cos \rho \\
& + \frac{\sin \rho}{2\rho^3}\int_0^1 \lambda(t)u(t)\left[e^{-\rho} \cos \rho(1 - t) - \frac{e^{-\rho t}}{2} - \frac{e^{-\rho(2-t)}}{2}\right] dt \\
& + \frac{\cos \rho}{2\rho^3}\int_0^1 \lambda(t)u(t)\left[- e^{-\rho} \sin \rho(1 - t) + \frac{e^{-\rho t}}{2} - \frac{e^{-\rho(2-t)}}{2}\right] dt \\
& + \frac{1}{4\rho^3}\int_0^1 \lambda(t)u(t)\,[\sin \rho(1 - t) - \cos \rho\,(1 - t) + e^{-\rho(1-t)}]dt \\
& - \frac{e^{-\rho}}{4\rho^3}\int_0^1 \lambda(t)u(t)\,[- e^{-\rho} \sin \rho(1 - t) - e^{-\rho} \cos \rho(1 - t) + e^{-\rho t}]dt.
\end{aligned}
$$

For each $\rho=\rho_n$, u is a continuous function; we show u uniformly bounded in n. In u, the terms

$$
- \frac{1}{4\rho^3}\int_0^1 \lambda(t)u(t)e^{\rho(x-t)}dt\big\{1 - e^{-\rho} \sin \rho - e^{-\rho} \cos \rho\big\}^{-1},
$$

$$
\frac{1}{4\rho^3}\int_0^x \lambda(t)u(t)e^{\rho(x-t)}dt
$$

may be combined to give

$$\left[-\frac{1}{4\rho^3}\int_x^1 \lambda(t)u(t)e^{\rho(x-t)}dt - \frac{\sin\rho+\cos\rho}{4\rho^3}\int_0^x \lambda(t)u(t)e^{-\rho(1-x+t)}dt\right]$$
$$\cdot\left\{1 - e^{-\rho}\sin\rho - e^{-\rho}\cos\rho\right\}^{-1}.$$

For t in $[x, 1]$, $x-t\leqq 0$; for t in $[0, x]$, $1-x+t\geqq 0$. Also, there exists a constant $c>0$ such that $1-e^{-\rho}\sin\rho-e^{-\rho}\cos\rho\geqq c$ for $\rho=\rho_n$. Call $M_n=\max_{[0,1]}|u_n(x)|$, $K=\int_0^1|\lambda(t)|dt$. Then

$$|u_n(x)| \leqq 2+\frac{5}{c}+\frac{17M_nK}{4c\rho_n^3}, \qquad M_n \leqq 2+\frac{5}{c}+\frac{17M_nK}{4c\rho_n^3},$$
$$M_n \leqq \frac{2+5/c}{1-17K/(4c\rho_n^3)}.$$

Thus for n sufficiently great, $M_n\leqq 4+10/c$. The remaining n's form a finite set. Hence $u_n(x)$ is uniformly bounded in n. Thus

$$u_n(x) = \cos\rho_n x - \sin\rho_n x + e^{-\rho_n(1-x)}\sin\rho_n - e^{-\rho_n x} + r_n(x)/\rho_n^3,$$

and $\cos\rho_n=\phi(\rho_n)$, where $\lim_{n\to\infty}\phi(\rho_n)=0$. Thus $\rho_n=(n+1/2)\pi+\epsilon_n$, where $\lim_{n\to\infty}\epsilon_n=0$. It follows that

$$\begin{aligned} u_n(x) = {} & \cos(n+1/2)\pi x - \sin(n+1/2)\pi x \\ & + (-1)^n\exp\{-(n+1/2)\pi(1-x)\} \\ & - \exp\{-(n+1/2)\pi x\} + r_n(x)/n^3. \end{aligned}$$

The theorem for system V follows.

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