A DECOMPOSITION OF ADDITIVE SET FUNCTIONS[1]

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This paper is concerned with a decomposition theorem for additive functions on an additive family of sets to either real numbers or a Banach space. Additive bounded set functions have as yet been little studied. However the recent paper of Hildebrandt[2] illustrates their importance.

We shall use the following notation:

(a) T: an abstract class of arbitrary elements t.

(b) $\mathfrak{T}$: a completely additive family of subsets τ of T; that is, $T \in \mathfrak{T}$, $\tau \in \mathfrak{T}$ implies $T-\tau \in \mathfrak{T}$, and $\tau_n \in \mathfrak{T}$ for $n=1, 2, \cdots$ implies $\sum\tau_n \in \mathfrak{T}$.

(c) α: a set function on $\mathfrak{T}$ to real numbers.

(d) A: the subclass of set functions on $\mathfrak{T}$ to real numbers which are additive and bounded; that is, $\tau_1, \tau_2 \in \mathfrak{T}$ and $\tau_1\cdot\tau_2=0$ implies $\alpha(\tau_1+\tau_2)=\alpha(\tau_1)+\alpha(\tau_2)$.

(e) C: the subclass of set functions on $\mathfrak{T}$ to real numbers which are completely additive (c.a.), that is, $\tau_n \in \mathfrak{T}$ for $n=1, 2, \cdots$ and $\tau_i\cdot\tau_j=0$ if $i\neq j$ implies $\alpha(\sum\tau_n)=\sum\alpha(\tau_n)$. The functions in C are bounded.[3]

The notations A_P and C_P refer to the subclasses of A and C respectively whose elements are nonnegative.

(f) x: a set function on $\mathfrak{T}$ to a Banach space[4] X. The definitions of additive and c.a. set functions are formally retained. If $\{\tau_n\}$ is a sequence of disjoint sets of $\mathfrak{T}$ and $x(\tau)$ is c.a., then $\sum x(\tau_n)$ is unconditionally convergent.[5]

(g) C_X: the class of c.a. set functions on $\mathfrak{T}$ to X.

In the statement of the following theorems, D will designate any one of the classes A, A_P, C, C_P, and $\bar{\tau}$ will denote the cardinal number of τ.

THEOREM 1. *Let $\aleph$ be an infinite cardinal number not greater than $\bar{\bar{T}}$. For every $\alpha \in D$ there exists an unique decomposition $\alpha=\alpha_1+\alpha_2$ and a set $R(\alpha) \in \mathfrak{T}$ of cardinal number not greater than $\aleph$ such that $\alpha_1, \alpha_2 \in D$,*

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[2] T. H. Hildebrandt, *On bounded linear functional operations*, Transactions of this Society, vol. 36 (1934), pp. 868–875.

[3] S. Saks, *Theory of the Integral*, Monografje Matematyczne, Warsaw, 1937, p. 10, Theorem 6.1.

[4] S. Banach, *Théorie des Opérations Linéaires*, Monografje Matematyczne, Warsaw, 1932, chap. 5.

[5] If x_n is a series of elements of X and if every subseries $\sum x_n$ is convergent, then $\sum x_n$ is said to be unconditionally convergent.

$$\alpha_1(\tau) = \alpha(R\cdot\tau), \qquad \alpha_2(\tau) = 0 \quad \textit{if} \quad \bar{\bar{\tau}} \leqq \aleph .$$

Let $\Sigma \equiv E_\tau[\tau \; \varepsilon \; \mathfrak{T}, \bar{\bar{\tau}} \leqq \aleph, \alpha(\tau) \neq 0]$. We define a transfinite sequence $(\tau_1, \tau_2, \cdots ; \tau_\omega, \cdots, \tau_\lambda, \cdots)$ as follows: τ_1 is an arbitrary element of Σ. Suppose τ_λ have been defined for all $\lambda < \mu$. If there exists τ such that $\tau \cdot \sum_{\lambda<\mu}\tau_\lambda = 0$ and $\tau \; \varepsilon \; \Sigma$, then we set $\tau = \tau_\mu$.

As $\alpha(\tau)$ is bounded, $\alpha(\tau)$ cannot differ from zero on a nondenumerable number of disjoint sets. The sequence therefore contains only a denumerable set of elements.

Let $R = \sum_\lambda \tau_\lambda$. Then $R \; \varepsilon \; \mathfrak{T}$ and $\bar{\bar{R}} \leqq \aleph$. We define $\alpha_1(\tau) = \alpha(R\cdot\tau)$, $\alpha_2(\tau) = \alpha(\tau) - \alpha_1(\tau) = \alpha(\tau - R\cdot\tau)$. The $\alpha_1(\tau)$, $\alpha_2(\tau)$ are clearly elements of D. If $\bar{\bar{\tau}} \leqq \aleph$, then by the definition of R, $\alpha_2(\tau) = \alpha(\tau - R\cdot\tau) = 0$.

Although the set R is not unique, the function decomposition is unique: Suppose there exist two different sets R_1, R_2 having the properties of the R defined above. The set identity $R_1\cdot\tau + (R_2 - R_1)\cdot\tau = R_2\cdot\tau + (R_1 - R_2)\cdot\tau$ and $\alpha[(R_1 - R_2)\cdot\tau] = 0 = \alpha[(R_2 - R_1)\cdot\tau]$ imply that $\alpha(R_1\cdot\tau) = \alpha(R_2\cdot\tau)$.

A set function α on $\mathfrak{T}$ will be said to be *nonsingular* if for every $t \; \varepsilon \; \mathfrak{T}$, $\alpha(t) = 0$. A set function α on $\mathfrak{T}$ will be called $\aleph$*-homogeneous* if there exists a set R such that $R \; \varepsilon \; \mathfrak{T}$, $\bar{\bar{R}} = \aleph$, $\alpha(\tau) = \alpha(R\cdot\tau)$, and $\alpha(\tau) = 0$ if $\bar{\bar{\tau}} < \aleph$.

Without loss of generality we may consider only nonsingular set functions because for every $\alpha \; \varepsilon \; D$ there exists a unique decomposition $\alpha = \alpha_1 + \alpha_2$ and a denumerable set $\{t_i\}$ of elements of T, such that $\alpha_1, \alpha_2 \; \varepsilon \; D$, $\alpha_1(\tau) = \sum_{i=1}^{\infty} \alpha(\tau\cdot t_i)$, and α_2 is nonsingular. We omit the proof.

THEOREM 2. *For every nonsingular $\alpha \; \varepsilon \; D$, there exists an unique decomposition $\alpha = \sum_i \alpha_i$, the sum being absolutely convergent, and such that α_i is $\aleph_i$-homogeneous and $\aleph_i \neq \aleph_j$ if $i \neq j$.*

In the proof of this theorem an induction is made on the infinite cardinals not exceeding that of T, well-ordered according to magnitude. We define a transfinite sequence of set functions $(\alpha_1, \alpha_2, \cdots ; \alpha_\omega, \cdots, \alpha_\lambda, \cdots)$ as follows: Suppose α_λ have been defined for all $\lambda < \mu$ and (1) only a denumerable number of the α_λ are not identically zero; (2) $\sum_{\lambda \leqq \lambda_0} |\alpha_\lambda(\tau)| < \infty$; and (3) $\alpha_\lambda \; \varepsilon \; D$ and is $\aleph_\lambda$-homogeneous. By Theorem 1 there exist $R_\mu \; \varepsilon \; \mathfrak{T}$ and a decomposition $\alpha = \alpha_\mu^1 + \alpha_\mu^2$ such that $\bar{\bar{R}}_\mu \leqq \aleph_\mu$, $\alpha_\mu^1(\tau) = \alpha(R_\mu\cdot\tau)$, $\alpha_\mu^2(\tau) = 0$ if $\bar{\bar{\tau}} \leqq \aleph_\mu$, and $\alpha_\mu^1, \alpha_\mu^2 \; \varepsilon \; D$. Clearly $\alpha_\lambda(\tau) = \alpha(R_\mu \cdot R_\lambda \cdot \tau)$ if $\lambda < \mu$.

Let $\alpha_\mu(\tau) = \alpha_\mu^1(\tau) - \sum_{\lambda<\mu}\alpha_\lambda(\tau)$. We consider the following cases:

I. $\alpha \; \varepsilon \; C, C_P$. Let $\bar{R}_\mu = R_\mu - \sum_{\pi_\mu} \bar{R}_\lambda$ where $\pi_\mu \equiv E_\lambda[\lambda < \mu, \alpha_\lambda \neq 0]$. The sets $\bar{R}_\mu$ are disjoint. Suppose $\alpha_\lambda(\tau) = \alpha(\bar{R}_\lambda\cdot\tau)$ for $\lambda < \mu$. Then by (1)

$$\alpha_\mu(\tau) = \alpha(R_\mu\cdot\tau) - \sum_{\pi_\mu} \alpha_\lambda(\tau) = \alpha(R_\mu\cdot\tau) - \sum_{\pi_\mu} \alpha(R_\mu\cdot\overline{R}_\lambda\cdot\tau)$$
$$= \alpha\left[\left(R_\mu - \sum_{\pi_\mu} R_\mu\cdot\overline{R}_\lambda\right)\cdot\tau\right] = \alpha(\overline{R}_\mu\cdot\tau).$$

It is clear that (1), (2), and (3) are satisfied for $\mu+1$. $\alpha_\lambda\neq 0$ implies that $\alpha(\tau)\neq 0$ for some subset of $\overline{R}_\lambda$. As the $\overline{R}_\lambda$ are disjoint, the sequence will contain only a denumerable number of functions not identically zero.

II. $\alpha \varepsilon A_P$. For $\lambda_0<\mu$, $\alpha(T)\geqq \overset{1}{\alpha}_{\lambda_0}(T)\equiv\sum_{\lambda\leqq\lambda_0}\alpha_\lambda(T)\geqq\sum_{\lambda\leqq\lambda_0}\alpha_\lambda(\tau)$. Clearly (1) and (2) are satisfied for $\mu+1$, and the sequence contains only a denumerable number of functions not identically zero. Let λ_i be a spanning sequence for $E_\lambda[\lambda<\mu, \alpha_\lambda\neq 0]$. Then

$$\alpha_\mu(\tau) = \overset{1}{\alpha}_\mu(\tau) - \sum_{\lambda<\mu} \alpha_\lambda(\tau) = \alpha(R_\mu\cdot\tau) - \lim_{i\to\infty} \overset{1}{\alpha}_{\lambda_i}(\tau)$$
$$= \alpha(R_\mu\cdot\tau) - \lim_{i\to\infty} \alpha(R_\mu\cdot R_{\lambda_i}\cdot\tau).$$

Hence (3) is likewise satisfied.

III. $\alpha \varepsilon A$. Every $\alpha \varepsilon A$ has a decomposition $\alpha=\alpha_1-\alpha_2$ where $\alpha_1, \alpha_2 \varepsilon A_P$. An application of II to α_1 and α_2 gives the desired decomposition.

The decomposition is unique: Any two sequences of homogeneous functions differ in a first function, α_μ. But this is contrary to $\overset{1}{\alpha}_\mu=\sum_{\lambda\leqq\mu}\alpha_\lambda$ being unique.

In these theorems the restriction that the additive bounded set function be defined over an additive family $\mathfrak{J}$ is optional, since the range of definition of such a function can always be extended to an additive family. The type of argument used by Pettis[6] will prove this statement.

We next consider the possibility of extending these theorems to functions $x(\tau)$ on $\mathfrak{J}$ to a Banach space. The theorem is not in general valid for additive bounded set functions of this type. This is illustrated by $x(\tau)$ defined on all subsets of $T\equiv(0, 1)$ to the space X of bounded functions on $S\equiv(0, 1)$ where $x(\tau)$ is the characteristic function of the subset of S which has the same coordinate values as τ. Clearly there exists no denumerable set R such that $x(\tau-R\tau)=0$ for all denumerable sets τ.

However analogous theorems are obtained for c.a. set functions on $\mathfrak{J}$ to X.

[6] B. J. Pettis, *Linear functionals and completely additive set functions*, Duke Mathematical Journal, vol. 4 (1938), p. 554, Theorem 1.1.

THEOREM 3. *Let* $\aleph$ *be an infinite cardinal number not greater than* $\overline{\overline{T}}$. *For every* $x \in C_X$ *there exists a unique decomposition* $x = x_1 + x_2$ *and a set* $R(x) \in \mathfrak{J}$ *of cardinal power not greater than* $\aleph$ *such that* $x_1, x_2 \in C_X$, $x_1(\tau) = x(R \cdot \tau)$, $x_2(\tau) = 0$ *if* $\overline{\overline{\tau}} \leqq \aleph$.

$x(\tau) \neq 0$ on at most a denumerable number of disjoint sets of $\mathfrak{J}$. Suppose the contrary. Then there exists a denumerable sequence of disjoint sets $\{\tau_i\}$ and an $e > 0$ such that $\|x(\tau_i)\| > e$, $(i = 1, 2, \cdots)$. As $x(\tau)$ is c.a., $\sum_i x(\tau_i)$ converges. The supposition is therefore false.

The argument used in Theorem 1 will now prove the theorem.

THEOREM 4. *For every nonsingular* $x \in C_X$, *there exists an unique decomposition* $x = \sum_i x_i$, *the sum being unconditionally convergent, and such that* x_i *is* $\aleph_i$*-homogeneous and* $\aleph_i \neq \aleph_j$ *if* $i \neq j$.

The proof is identical with that of I in Theorem 2. Again there will exist disjoint $\overline{R}_\mu$'s such that $x_\mu(\tau) = x(\overline{R}_\mu \cdot \tau)$.

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