INDEFINITELY DIFFERENTIABLE FUNCTIONS WITH PRESCRIBED LEAST UPPER BOUNDS[1]

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1. Introduction. Let $F(x)$ be a real indefinitely differentiable function of the real variable x defined on the interval $a \leqq x \leqq b$, and let M_n denote the least upper bound of $|F^{(n)}(x)|$ on that interval.[2] In this paper we shall establish sufficient conditions that there exist an indefinitely differentiable function $F(x)$ taking on certain prescribed M_n.

It is easy to see that M_0 and M_1 can be assigned arbitrarily. However, the first three upper bounds M_0, M_1, and M_2 must satisfy certain inequalities.[3] Let us consider the interval (0, 1). Let t_1 be the value of x for which $|F^{(1)}(x)|$ attains its maximum. Then

$$F(1) - F(t_1) = (1 - t_1)F^{(1)}(t_1) + (1/2!)F^{(2)}(\theta_1)(1 - t_1)^2,$$

where $t_1 < \theta_1 < 1$. And similarly

$$F(0) - F(t_1) = - t_1F^{(1)}(t_1) + (1/2!)F^{(2)}(\theta_2)t_1^2,$$

where $0 < \theta_2 < t_1$. On subtracting these equations we obtain

$$F^{(1)}(t_1) = F(1) - F(0) + (1/2!)\{F^{(2)}(\theta_2)t_1^2 - F^{(2)}(\theta_1)(1 - t_1)^2\},$$
$$M_1 \leqq 2M_0 + M_2/2!.$$

By the same procedure we can obtain for the interval $(0, a)$

$$M_1 \leqq 2M_0/a + M_2a/2!. \tag{1}$$

In the case of the interval $(0, \infty)$ we can replace (1) by a more precise inequality. Since a is arbitrary, we can replace a by the positive value which minimizes the right side of (1), and obtain

$$M_1 \leqq 2(M_0M_2)^{1/2}.$$

Ore[4] in a recent paper employed the results of W. Markoff to obtain certain inequalities connecting the least upper bounds of $|F^{(i)}(x)|$, $(1 \leqq i \leqq n)$, with those of $|F(x)|$ and $|F^{(n+1)}(x)|$ where $F(x)$ is a function with bounded $(n+1)$th derivative. For the first derivative the inequality (1) is slightly better than that obtained by Ore.

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[2] By $F^{(0)}(x)$ we shall mean $F(x)$.

[3] Hadamard, Comptes Rendus des Séances de la Société Mathématique de France, 1914, pp. 68–72; T. Carlman, *Les fonctions quasi analytiques*, Paris, 1926.

[4] O. Ore, *On functions with bounded derivatives*, Transactions of this Society, vol. 43 (1938), pp. 321–326.

2. **Construction of an indefinitely differentiable function with prescribed least upper bounds.** We now prove the following theorem.

THEOREM. *If the sequence* $\{M_i a^i\}$ *is monotone decreasing, then there exists an indefinitely differentiable function* $F(x)$ *defined on* $(0 \leqq x \leqq a)$ *such that* M_i *is the least upper bound of* $|F^{(i)}(x)|$ *on* $(0 \leqq x \leqq a)$.

Define

$$0 \leqq S_i = \sum_{j=0}^{\infty} (-1)^j \frac{M_{i+j}a^j}{j!} \leqq M_i, \qquad F(x) = \sum_{i=0}^{\infty} S_i \frac{x^i}{i!} \cdot$$

Now the function $F(x)$ so defined is an entire function. Let $x = b$. Then

$$F(b) = \sum_{i=0}^{\infty} S_i \frac{b^i}{i!} = \sum_{i=0}^{\infty} (-1)^i \frac{M_i}{i!}\left(\sum_{j=0}^{i} (-1)^j C_{i-j,j} a^{i-j} b^j \right),$$

$$= \sum_{i=0}^{\infty} (-1)^i \frac{M_i(a-b)^i}{i!} \cdot$$

And since the series

$$\sum_{i=0}^{\infty} \frac{M_i a^i |(1-b/a)^i|}{i!} < M_0 \sum_{i=0}^{\infty} \frac{|(1-b/a)^i|}{i!}$$

converges, the series $\sum_{i=0}^{\infty}(S_i/i!)b^i$ converges.[5] Further,

$$F^{(i)}(a) = \sum_{j=0}^{\infty} (-1)^j M_{i+j} a^j \left(\sum_{k=0}^{j} \frac{(-1)^k}{k!(j-k)!} \right) = M_i,$$

and

$$|F^{(i)}(x)| \leqq F^{(i)}(a) = M_i, \qquad 0 \leqq x \leqq a.$$

Thus we have given explicitly a function $F(x)$ satisfying the conditions of the theorem.

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[5] See T. J. I'A. Bromwich, *Theory of Infinite Series*, London, 1931, p. 266.