ANOMALOUS PLANE CURVE SYSTEMS ASSOCIATED WITH SINGULAR SURFACES[1]

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1. Introduction. A plane curve system of order n and genus p, with δ nodes, κ cusps, ι statangents, τ bitangents, is of first virtual dimension d_0, of virtual dimension d_0', and of effective dimension d. For such a system, $d_0' = 3n+p-\kappa-1 = 3m+p-\iota-1$. If the curve system has only distinct nodes and cusps, $d_0 = d_0'$; if higher singularities, $d_0 < d_0'$.

For any irreducible, continuous, complete curve system, $d \geqq d_0$. If $d = d_0$, the curve system is called homalous, if $d > d_0$, anomalous. The anomaly A of a plane curve system is defined by the relation $A = d - d_0$. The above definitions were introduced by B. Segre.[2]

The sections by a plane π of the tangent cones to a nonsingular surface of order ν from the points of S_3 constitute a continuous plane curve system with the characteristics:

$$n = \nu(\nu - 1), \qquad \kappa = \nu(\nu - 1)(\nu - 2),$$
$$\delta = (1/2)\nu(\nu - 1)(\nu - 2)(\nu - 3).$$

B. Segre[3] has shown that for this curve system,

$$d_0 = (5/2)\nu(\nu - 1), \qquad d = (1/6)(\nu + 1)(\nu + 2)(\nu + 3) - 5,$$
$$A = d - d_0 = (1/6)(\nu - 2)(\nu - 3)(\nu - 4).$$

Hence, for $\nu \geqq 5$, this plane curve system is anomalous.

The purpose of the present paper is to ascertain the dimensions and anomaly of plane sections of tangent cones to certain singular surfaces.

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[1] Presented to the Society, April 16, 1938.

[2] B. Segre, *Esistenza e dimensione di sistemi continui distinti di curve piane algebriche con dati caratteri*, Rendiconti dell'Accademia dei Lincei, (6), vol. 10 (1929), pp. 31–38. The adjective "irregolare" used by B. Segre to describe the plane curve system for which $d > d_0$ has two English equivalents "irregular" and "anomalous." The first translation is the more natural one. Since it has been found, however, that similar systems of surfaces exist and since the term "irregular" has long had a different definite meaning for surfaces, another term is necessary for surfaces and, therefore, should also be used for plane curves. Homalous and anomalous are Greek antonyms. Only the latter has heretofore been used in English.

[3] B. Segre, *Sulla caratterizzazione delle curve di diramazione dei piani multipli generali*, Memorie delle Reale Accademia d'Italia, Classe di Scienze Fisische, Matematiche e Naturali, vol. 1 (1930), pp. 5–31.

2. **Surfaces with multiple points.** Let the surface F of order ν have the point P of multiplicity k, such that the plane section of the tangent cone to F at P is a nonsingular curve of order k. The tangent cone to F from an arbitrary point has a multiple generator of order $k(k-1)$ through P. This multiple generator reduces the number of distinct cuspidal generators by $k(k-1)(k-2)$ and the number of distinct nodal generators by $(1/2)k(k-1)(k-2)(k-3)$.

The plane sections C by a plane π of the tangent cones to F from points of S_3 consist of plane curves each of which has a $k(k-1)$-fold point at the intersection of π with the multiple generator. This plane curve system C has the additional characteristics

$$
\begin{aligned}
n &= \nu(\nu-1), \qquad m = \nu(\nu-1)^2 - k(k-1)^2,\\
p &= (1/2)\nu(\nu-1)(2\nu-5) - (1/2)k(k-1)(2k-3),\\
\kappa &= \nu(\nu-1)(\nu-2) - k(k-1)(k-2),\\
\delta &= (1/2)\nu(\nu-1)(\nu-2)(\nu-3) - (1/2)k(k-1)(k-2)(k-3),\\
d_0 &= (5/2)\nu(\nu-1) - (3/2)k(k-1) + 2.
\end{aligned}
$$

In the above, δ and κ are the numbers of distinct nodes and cusps of the plane sections. The $k(k-1)$-fold point contains also the equivalent of $(1/2)k(k-1)(k^2-k-1)$ nodes.

To obtain the effective dimension of C, the effective dimension of F must be found. The invariant postulation[4] of a k-fold point on F is $(1/6)k(k+1)(k+2)-3$. Since the above plane curve system may be the plane section of the tangent cone to ∞^4 surfaces, the effective dimension d of C is

$$d = (1/6)(\nu+1)(\nu+2)(\nu+3) - (1/6)k(k+1)(k+2) - 2.$$

Hence

$$A = (1/6)(\nu-2)(\nu-3)(\nu-4) - (1/6)(k-1)(k-2)(k-3).$$

The above formulas are derived for one multiple point of order k. If the surface has α distinct, independent, multiple points of orders k_i, the above formulas hold with the sum sign. For example, for such a surface,

$$A = (1/6)(\nu-2)(\nu-3)(\nu-4) - (1/6)\sum_{i=1}^{\alpha}(k_i-1)(k_i-2)(k_i-3).$$

From the above expression for A, there results:

(1) A surface may have any number of independent double or

[4] T. R. Hollcroft, *Invariant postulation*, this Bulletin, vol. 36 (1930), p. 421.

triple points without affecting the anomaly of plane sections of the tangent cone.

(2) The plane sections of the tangent cones to a monoid ($\alpha=1$, $k=\nu-1$) are always homalous plane curve systems.

Since $A\neq 0$, this expression for A ($k_i=k$) defines the following limit to the number α of distinct independent k-fold points, $k\geqq 4$, on an irreducible surface of order ν:

$$\text{(A)} \qquad \alpha \leqq \frac{(\nu-2)(\nu-3)(\nu-4)}{(k-1)(k-2)(k-3)} .$$

The smallest limit obtained heretofore results from the fact that the genus of the plane section of the tangent cone to an irreducible surface cannot be negative. This limit[5] is

$$\text{(B)} \qquad \alpha \leqq \frac{(\nu-2)(2\nu^2-3\nu-1)}{k(k-1)(2k-3)} .$$

Limit (A) is smaller than limit (B) for a given value of $k\geqq 4$ and $\nu=k+\beta$, where

$$\beta \leqq \frac{3(6k-k^2-6)+(121k^4-832k^3+1988k^2-1920k+612)^{1/2}}{2(7k-12)} .$$

A very close approximation to this limiting value of β is given by $\beta\leqq(1/7)(4k-3)$.

It has been assumed that the multiple points are unrelated in position on the surface and that they are associated with sets of independent invariants among the coefficients of the surface.

The nodes and cusps of the system C lie on adjoint curves. The adjoint curves of the lowest order are the following:

(1) The adjoint curves of order $(\nu-1)(\nu-2)$ pass simply through the $\delta+\kappa$ distinct double points and have a multiple point of order $(k_i-1)(k_i-2)$ at each $k_i(k_i-1)$-fold point.

(2) The adjoint curves of order $\nu(\nu-2)$ pass simply through the nodes, are tangent to C at each cusp, and have a multiple point of order $k_i(k_i-2)$ at each $k_i(k_i-1)$-fold point.

If the multiple points have cones whose plane sections are singular curves, the values of d and d_0 can be found in a given case, but a general formula that will include all such cases is not feasible.

[5] T. R. Hollcroft, *Limits for multiple points and curves of surfaces*, Tôhoku Mathematical Journal, vol. 30 (1928), pp. 116–117. Limit (1) derived on page 116 depends upon the Lefschetz postulate and, therefore, does not hold.

3. **Surfaces with singular curves.** Consider the surface F of order ν and class ν' with a nodal curve of order b and a cuspidal curve of order c. The other surface characteristics to be used are the following:

a, order of tangent cone = class of plane section;
δ, number of nodal generators of tangent cone;
κ, number of cuspidal generators of tangent cone;
σ, class of cuspidal developable;
σ', order of parabolic curve;
j, number of pinch points on nodal curve;
ω, number of tacnodal points on cuspidal curve;
χ, number of cubic nodes on cuspidal curve;
Δ, effective dimension of surface.

From the Cayley-Zeuthen equations, there results

$$(1)\quad a = \nu(\nu - 1) - 2b - 3c, \quad 2\delta + 4\kappa = a(a - 2) + \sigma + 2j + 3\chi + 3\omega.$$

Since, for the plane sections C of the tangent cone,

$$d_0 = (1/2)a(a + 3) - \delta - 2\kappa,$$

there results, from the above,

$$(2)\qquad d_0 = (1/2)(5a - \sigma - 2j - 3\chi - 3\omega).$$

For a nonsingular surface, $d_0 = 5a/2$.

No such general formula exists for the effective dimension d of this curve system C. Further and more specific characteristics of the double curves must be given. For example, the dimension of a space curve may involve properties of the two surfaces of which it is the partial or complete intersection. In order to determine the effective dimension Δ of F, the intersection characteristics of the nodal and cuspidal curves must be known.

In particular cases, when sufficient data is given, Δ can be found, and therefore d. Nodal and cuspidal lines will be investigated.

4. **Surfaces with a nodal line.** The surface F of order ν has a nodal line both of whose tangent planes are torsal. The characteristics of F are

$$a = \nu(\nu - 1) - 2, \quad b = 1, \quad j = 2(\nu - 2), \quad \nu' = (\nu - 2)(\nu^2 - 6).$$

The virtual dimension of the plane section C of the tangent cone is $d_0 = (1/2)(5a - 2j) = (1/2)(\nu - 2)(5\nu + 1)$.

The postulation of a nodal line on F is $3\nu + 1$. Its invariant postulation[6] is $3(\nu - 1)$.

[6] *Invariant postulation*, loc. cit., p. 424.

Hence, for F

$$\Delta = (1/6)(\nu + 1)(\nu + 2)(\nu + 3) - 3\nu + 2.$$

The effective dimension of C is $d = \Delta - 4$, hence

$$d = (1/6)(\nu + 1)(\nu + 2)(\nu + 3) - 3\nu - 2.$$

The anomaly of the system C is

$$A = d - d_0 = (1/6)\nu(\nu - 4)(\nu - 5).$$

If one tangent plane of the nodal line is fixed and the other torsal, the line has $2(\nu - 2)$ pinch points that coincide in pairs. The computation is the same as above for both planes torsal.

5. **Surfaces with a cuspidal line.** A surface F may have two types of cuspidal lines, (1) the tangent plane to F is fixed, that is, is the same for all points of the line; (2) the tangent plane to F along the line is torsal.

(1) The surface F of order ν has a cuspidal line, such that the same plane is tangent to F at every point of this line. F has the characteristics

$$a = \nu(\nu - 1) - 3, \quad \nu' = (\nu - 3)(\nu^2 + \nu - 8),$$
$$\omega = \nu - 2, \quad \chi = \nu - 3, \quad \kappa = (\nu - 3)(\nu^2 - 4), \quad d_0 = (1/2)\nu(5\nu - 11),$$
$$d = \Delta - 4 = (1/6)(\nu + 1)(\nu + 2)(\nu + 3) - 4\nu - 1,$$
$$A = (1/6)\nu(\nu - 4)(\nu - 5).$$

(2) The surface F of order ν with a cuspidal line whose tangent plane is torsal has the characteristics

$$a = \nu(\nu - 1) - 3, \quad \nu' = (\nu - 3)(\nu^2 + \nu - 8),$$
$$\chi = \nu, \quad \omega = \nu - 4, \quad \sigma = 1, \quad \kappa = (\nu - 3)(\nu^2 - 4) + 4,$$
$$d_0 = (1/2)\nu(5\nu - 11) - 2,$$
$$d = \Delta - 4 = (1/6)(\nu + 1)(\nu + 2)(\nu + 3) - 4\nu - 3,$$
$$A = (1/6)\nu(\nu - 4)(\nu - 5).$$

In this case, the cuspidal developable is a point, and is, therefore, of order zero and class $\sigma = 1$.

Although the values of certain characteristics are different, the value of A is the same for both types of cuspidal lines.

6. **Summary for double lines.** The results of the two preceding sections lead to the following conclusion:

If an algebraic surface F of order $\nu \geqq 4$ has a double line (either nodal or cuspidal), the anomaly A of the plane sections C of a tangent cone to F is $A=(1/6)\nu(\nu-4)(\nu-5)$. As is evident from this expression for A, plane curve systems C associated with a surface having a double line are anomalous only for $\nu \geqq 6$.

When $\nu=3$, the above formula yields $A=1$. However, a cubic surface with a nodal line is ruled, and a cubic surface with a cuspidal line is a cone. The treatment given in the two preceding sections and the resulting expression for A do not apply to such surfaces.

WELLS COLLEGE

ON THE STABILITY OF THE LAMINAR FLOW OF A VISCOUS FLUID[1]

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The problem of the effect of two-dimensional first-order disturbances upon the laminar flow of an incompressible viscous fluid is known to be fundamental to the analysis of the phenomenon of turbulence. This discussion is concerned with such a problem in the case of a flow which takes place parallel to and between parallel plane boundaries. If the direction normal to these boundaries is designed by y, and that of the flow by x, the unit of length and the origin of coordinates may be chosen so that the boundary planes are given by the equations $y=1$ and $y=-1$. It is to be assumed then that the velocity of the undisturbed flow depends only on y, and is given by a function $U(y)$, which is suitably differentiable and nonnegative, which is nonincreasing as to $|y|$, and which vanishes at the boundaries. If the maximum velocity is chosen as the unit, as will be assumed, it follows that

$$U(0) = 1, \qquad U(1) = 0, \qquad U(-y) \equiv U(y).$$

The disturbance imposed upon this flow is to be taken as an elementary wave of length $2\pi/\alpha$, in the direction of flow.

The problem as stated is known[2] to admit of formulation as the differential boundary problem

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[1] Presented to the Society, September 7, 1939.

[2] C. L. Pekeris, *On the stability problem in hydrodynamics*, I, Proceedings of the Cambridge Philosophical Society, vol. 32 (1936), p. 55, and II, Journal of the Aeronautical Sciences, vol. 5 (1938), p. 236.