NOTE ON A PRINCIPAL AXIS TRANSFORMATION FOR NON-HERMITIAN MATRICES

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In a recent note the following theorem was proved.†

THEOREM 1. *If A is a matrix, over the complex field, of r rows and s columns, there exist two unitary matrices U and V, such that*

$$UAV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is a real diagonal matrix $[d_1, d_2, \cdots, d_k]$ with positive elements d_i.

For completeness the following result may be added: *The elements d_i are determined uniquely as the positive square roots of the nonzero characteristic roots of the positive hermitian matrix AA^*, where A^* is the conjugate transposed of A.*

The elements d_i thus form a complete set of invariants for the matrix A under such unitary transformations. This, together with the theorem itself, may be proved as follows. If $r \leqq s$ and k is the rank of A, $A = (P, 0)V$, where V is unitary and P is the positive hermitian matrix of order r and rank k uniquely determined‡ by the equation $AA^* = P^2$. The hermitian matrix P is unitarily equivalent to the diagonal matrix of order r, whose first k elements are d_i, $(i = 1, 2, \cdots, k)$. In case $r \geqq s$, the polar representation

$$A = U\begin{pmatrix} P \\ 0 \end{pmatrix}$$

may be used, to prove the desired result.

The following theorem may also be of interest.

THEOREM 2. *Let A and B be two matrices, over the complex field, of r rows and s columns. Necessary and sufficient conditions that there exist two unitary matrices U and V, such that*

$$UAV = A_1, \qquad UBV = B_1, \tag{1}$$

where A_1 and B_1 are diagonal matrices, are that

† Carl Eckart and Gale Young, *A principal axis transformation for non-hermitian matrices*, this Bulletin, vol. 52 (1939), pp. 118–121.

‡ John Williamson, *A polar representation of singular matrices*, this Bulletin, vol. 41 (1935), pp. 118–123.

(2) $$AB^* = f(BA^*), \qquad B^*A = f(A^*B),$$

where $f(x)$ is a polynomial in x.

We first prove the necessity of the conditions. Since $A_1B_1^*$ is a diagonal matrix, $A_1B_1^*$ is normal† and therefore $A_1B_1^*=f(B_1A_1^*)$. Since A_1 and B_1^* are both diagonal matrices, $A_1B_1^*$ coincides with $B_1^*A_1$ except for zero elements and so does $B_1A_1^*$ with $A_1^*B_1$. Therefore $B_1^*A_1=f(A_1^*B_1)$ and, as a consequence of (1), (2) is true.

To prove the sufficiency we note that there is no loss in generality in assuming that A is of the form

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is a real nonsingular diagonal matrix. Let

$$\begin{pmatrix} H & L \\ M & K \end{pmatrix}$$

be a partition of B similar to that of A. Since AB^* is normal, $AB^*BA = BAAB^*$ and by a simple calculation we find that

$$MDDM^* = (MD)(MD)^* = 0.$$

Hence $MD=0$ and, since D is nonsingular, $M=0$. In a similar manner by using the fact that B^*A is normal we find that L is zero. Conditions (2) now reduce to

(3) $$DH^* = f(HD), \qquad H^*D = f(DH).$$

Since $f(HD)=D^{-1}f(DH)D$, it follows from (3) that $D^2H^*=H^*D^2$. But D is a positive definite real matrix. Hence $DH^*=H^*D$ and consequently $DH=HD$. Since DH^* is normal, H^* and, therefore H, is normal. Since H is normal‡ and commutative with D, there exists a unitary matrix U_1, such that $U_1DU_1^*=D$ and $U_1HU_1^*=D_1$, where D_1 is diagonal but not necessarily real. By Theorem 1 there exist two unitary matrices U_2 and V_2, such that $U_2KV_2=D_2$, where D_2 is diagonal and real. If

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}, \qquad V = \begin{pmatrix} U_1^* & 0 \\ 0 & V_2 \end{pmatrix},$$

then

† Aurel Wintner, *Spektraltheorie der unendlichen Matrizen*, Leipzig, 1929, p. 24; John Williamson, *Matrices normal with respect to an hermitian matrix*, American Journal of Mathematics, vol. 60 (1938), p. 355.

‡ Aurel Wintner, op. cit., p. 24.

$$UAV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \qquad UBV = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

Since the elements of D and D_2 are real, we have the following corollary.

COROLLARY 1. *If k is the smaller of the ranks of A and B, A_1 and B_1 may be so determined that at most k of the elements of A_1 and B_1 are not real.*

If $f(x) = x$, AB^* and B^*A are both hermitian and D_1 must be real, so that B_1 is real. Conversely, if B_1 is real, $A_1B_1^*$ and $B_1^*A_1$ are both hermitian and therefore AB^* and B^*A are both hermitian. This result gives Theorem 2 of the note quoted in the first footnote. If $f(x) = 1/x$, AB^* and B^*A are both unitary and the elements of D_1 are $d_j^{-1}e^{i\theta_j}$.

When $r = s$, so that the matrices A and B are square, condition (2) may be replaced by the condition that AB^* *be normal and unitarily equivalent to* B^*A. For AB^* is similar to B^*A and, since both are normal, AB^* is unitarily equivalent to B^*A.

Both Theorems 1 and 2 are true in the real field, if A^* denotes the transposed of A and "unitary" is replaced by "orthogonal," except that D_1 may have two-rowed matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

in its diagonal.† In the simplest case, when AB^* and B^*A are both symmetric, D_1 will necessarily be diagonal, and Theorem 2 of the paper, quoted in the first footnote, is true in the real field, if "unitary" is replaced by "orthogonal" and "hermitian" by "symmetric."

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† F. D. Murnaghan and Aurel Wintner, *A canonical form for real matrices under orthogonal transformations*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 417–420.