$$\sigma_0 \leqq \limsup_{n\to\infty} (2 \log n)/\nu_n.$$

In order that $\sigma_0 \leqq 0$, in which case (3.1) will converge in the right half of the s-plane, it is sufficient that ν_n tend to infinity faster than log n. The argument used to complete the proof of Theorem 2 is the same as the one used above in connection with Theorem 1.

Notice that if $\{\nu_{n+1}-\nu_n\}$ is not a null sequence, then ν_n tends to infinity faster than log n. This eliminates the extra restriction used in the proof of Theorem 1.

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ON CERTAIN IDEALS OF DIFFERENTIAL POLYNOMIALS*

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Introduction. Let Σ be an ideal of differential polynomials in the unknowns $y_1, \cdots, y_n$. If the manifold of Σ is composed of s manifolds $\mathfrak{M}_1, \cdots, \mathfrak{M}_s$ not necessarily irreducible, none of which has a solution in common with any other, Σ has a unique representation as the product of s ideals $\Sigma_1, \cdots, \Sigma_s$ whose manifolds are, respectively, the $\mathfrak{M}_i$.†

Most of the present note is concerned with decompositions of the foregoing type and considers the case in which one of the $\mathfrak{M}_i$, say $\mathfrak{M}_1$, is composed of a single solution, that is, of a set of functions $\bar{y}_1, \cdots, \bar{y}_n$ contained in the underlying field. We shall examine, for this special case, the structure of the ideal Σ_1. Details will be given only for the case of a single unknown; the extensions to several unknowns are too obvious to require explicit mention. It will suffice, furthermore, to treat the case in which $\mathfrak{M}_1$ is composed of the solution $y=0$.

In §9, we consider a problem closely related to the theorem of decomposition stated above.

1. **On the structure of** Σ_1. Let Σ be an ideal of forms in the unknown y. Let $y=0$ be an essential irreducible manifold for Σ. Let Σ be the product of Σ_1 and Σ_2 where Σ_1 has $y=0$ as its manifold and Σ_2 does not admit $y=0$ as a solution. Let p be a positive integer such that y^p is contained in Σ_1.

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† Proceedings of the National Academy of Sciences, vol. 25 (1939), p. 90. *Product* is defined in the expected way. That the intersection of the Σ_i is identical with their product follows immediately from the fact that the Σ_i, considered as *algebraic* ideals, are *paarweise teilerfremd*. See van der Waerden, *Moderne Algebra*, vol. 2, p. 46.

We shall prove that Σ_1 *is the ideal generated by* Σ *and* y^p, that is, the intersection of all ideals containing Σ and y^p.

PROOF. Obviously Σ_1 contains the ideal generated as above. What has to be proved is the converse of this fact. Let G be any form in Σ_1. Σ_2 contains a form $1-H$, where H vanishes for $y=0$. Then Σ contains $G(1-H)$ and, therefore, $G(1-H^q)$ where q is any positive integer. Now, if q is large,

$$H^q \equiv 0, \ (y^p). \tag{1}$$

Let q be fixed at a value large enough for (1) to hold, and let $M=G(1-H^q)$. Then $G\equiv M$, (y^p), and this establishes the theorem.

2. **Condition for p to be unity.** We are going to examine now the case in which Σ contains a form of the type $y+A$, where A, considered as a polynomial in the y_j, has no term of degree less than 2. Ideals of this type form a natural and interesting class; a very special example is the ideal generated by y_1^2-4y. We are going to prove that the p of §1 may be taken as unity. That is, Σ_1 *consists of all forms which vanish for* $y=0$.

What we have to prove, of course, is that Σ_1 contains y. Our procedure will be as follows. For some p, y^p is in Σ_1. Then, for any i, $y_i^{2^ip}$ is in Σ_1. Let $F=y+A$ be the form mentioned in the statement of our theorem. Then

$$y \equiv -A, \ (F). \tag{2}$$

We shall subject (2) to an iterative process and derive a relation $y\equiv K$, (F), where every term of K contains some $y_i^{2^ip}$ as a factor, i depending on the term. This will establish the theorem.

3. **Bound on degrees.** Let a form P in y be of degree g in some y_j, $j\geqq 0$. We shall show that P', the derivative of P, has a degree in y_j which does not exceed $g+1$. For let L, any term of P, be divisible by y_j^q with $q\leqq g$, and by no higher power of y. Let $L=y_j^qM$. We have, indicating first derivatives by an accent,

$$L' = qy_j^{q-1}y_{j+1}M + y_j^qM'.$$

M' consists of a set of terms, one of which will be divisible by the first power of y_j if M involves y_{j-1}. This is enough to prove our statement.

4. **The first substitution.** Let us suppose that, in addition to (2), we have a second relation $y\equiv B$, (F). In the second member of (2), let y be replaced by B and each y_j by the jth derivative of B. Then $-A$ goes over into a form C. It is easy to see that $y\equiv C$, (F).

Let r be a positive integer which is not less than the order of A in y. Let g be an integer, exceeding unity, such that each term of A is of total degree not less than g in $y, \cdots, y_r$. In A, we replace y_j by $-A^{(j)}$, $j=0, \cdots, r$, superscripts indicating differentiation.* Then $-A$ goes over into a form A_1 and $y \equiv A_1$, (F). Each term in A_1 is of total degree not less than g^2 in $y, \cdots, y_{2r}$. By §3, the $A^{(j)}, j \leqq r$, are of degree not greater than r in any one of the letters $y_{r+1}, \cdots, y_{2r}$.

Let L be a term in A_1, of total degree $d \geqq g^2$ in the y_j. The power product of degree d in L is the product of a set of power products taken from the $A^{(j)}$. If M is any of the latter power products, the total degree of M is at least g, hence at least g/r times the degree of M in any one of $y_{r+1}, \cdots, y_{2r}$. Thus, the degree of L in any one of $y_{r+1}, \cdots, y_{2r}$ is not more than $(rd)/g$.

5. **The second substitution.** Differentiating A_1, we consider the $A_1^{(j)}$ for $j=0, \cdots, r$. No $A_1^{(j)}$ is of degree exceeding r in any y_i with $2r < i \leqq 3r$. Let L, of some total degree d, be a term in some $A_1^{(j)}$. Then L, since it is derived from a term of total degree d in A_1, is of degree not exceeding $rdg^{-1}+r$ in any y_i with $r < i \leqq 2r$. As $d \geqq g^2$, we have $rdg^{-1}+r \leqq rd(g^{-1}+g^{-2})$.

In the second member of (2), we replace each y_j by $A_1^{(j)}$. We find a relation $y \equiv A_2$, (F), with each term of A_2 of total degree at least g^3. If L is a term in A_2, of some total degree d, the degree of L in any y_i with $2r < i \leqq 3r$ does not exceed rdg^{-2} and the degree of L in any y_i with $r < i \leqq 2r$ does not exceed $rd(g^{-1}+g^{-2})$.

6. **Continuation.** In the third step, we substitute the $A_2^{(j)}$ into (2). After t steps, we have $y \equiv A_t$, (F), with each term in A_t of total degree at least g^{t+1}. Let L be a term in A_t of some total degree d. Let j be any positive integer not greater than t. Then the degree of L in any y_i with $jr < i \leqq (j+1)r$ does not exceed

$$rd(g^{-j} + g^{-j-1} + \cdots + g^{-t}) < 2rdg^{-j}.$$

7. **Completion of proof.** Let t of §6 be the square of a positive integer s. The total degree of L of §6 in the y_i with $i > sr$ is no more than

$$2rd(rg^{-s} + \cdots + rg^{-t}) < 4r^2dg^{-s}.$$

Let s be so great that

$$4r^2g^{-s} < 1/2.$$

Then the total degree of L in the y_i with $i \leqq sr$ is at least $d/2$. Thus, for some particular y_i with $i \leqq sr$, the degree of L in y_i is at least

* $A^{(0)} = A$.

$$\frac{d}{2(rs+1)} \geqq \frac{g^{s+1}}{2(rs+1)}. \tag{3}$$

We refer now to §2. If s is large, the second member of (3), if $i \leqq sr$, will exceed $2^i p$. This completes the proof of our theorem.

8. **Higher values of p.** It is not an unnatural conjecture that, if Σ contains a form $y^n + A$ with every term in A of degree greater than n, p of §1 may be taken as n. We give an example to show that the least p may exceed n.

Let Σ be the ideal generated by $F = y^3 + y_1^4$. If Σ_1 contained y^3, there would exist a relation

$$y^3(1 - H) = MF + M_1F' + \cdots + M_rF^{(r)}, \tag{4}$$

with H vanishing for $y = 0$. For the second member of (4) to yield the term y^3 which the first member contains, it would be necessary for M to have unity as one of its terms. Then MF would have y_1^4 as a term. Equating terms of degree 4 and weight 4 for both sides of (4), we would find $y_1^4 \equiv 0$, (y^3), which is readily shown to be false.

9. **A generalization.** Let F and A be two forms in $y_1, \cdots, y_n$, both of class n and algebraically irreducible. Suppose that the general solution $\mathfrak{M}$ of A is contained in the manifold of F and is essential in that manifold. It is known how the essentiality of $\mathfrak{M}$ is reflected in the structure of F.* Suppose now that S^tF, where S^t is as in the indicated theorem of structure, has a term C_jA. It can be shown, by the method of §§2–7 above, that *there exists a relation* $AH \equiv 0$, (F), *where* H *does not hold* $\mathfrak{M}$.

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* American Journal of Mathematics, vol. 60 (1938), p. 14.