AN EXAMPLE IN THE THEORY OF POINTWISE PERIODIC HOMEOMORPHISMS*

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A single-valued continuous transformation $T(M) = M$ is said to be *pointwise periodic*‡ provided that for every point x in M there exists an integer N_x such that $T^{N_x}(x) = x$. The smallest such integer N_x (greater than zero) is called the *period* of x under T. For each point x of M the finite subset of M consisting of all the images of x under T is called the orbit of x under T.

In most of the familiar examples concerned with this type of transformation the limit§ of every convergent sequence of orbits consists of a set of points all having the same period. In fact the first example in which this is not the case has recently been given by G. E. Schweigert and the author.‖ In this paper we combine this example with one found by Ralph Phillips and W. L. Ayres¶ and then generalize the result to obtain what is probably the first example in which the space is a locally connected continuum while at the same time there exists a sequence of orbits converging to a set L containing points of arbitrary preassigned periods.

We let $r_1 = 1$ and $r_2, r_3, \cdots$ be the arbitrarily chosen periods for points in L. Then we construct a locally connected continuum M in four-dimensional euclidean (x, y, z, w) space and define a pointwise periodic homeomorphism $T(M) = M$ with the following properties:

(a) There exists a convergent sequence of orbits (G_i) under T having a limit set L which for every $i = 1, 2, \cdots$ contains a free arc every interior point of which has period exactly r_i.

(b) The closure of every component of $M - L$ is a 2-cell.

We begin by constructing the limit set L in the (x, y) plane, and do this by means of polar coordinates (r, θ). In this plane let C be the unit circle $r=1$, and define an infinite sequence of points (a_i) as follows: $a_0 = (1, 0)$, $a_i = (1, [1+(i-1)/i]\pi)$, $(i \neq 0)$. Evidently, $\lim a_i = a_0$.

If the sequence (r_i), $(i=1, 2, \cdots)$, contains only a finite number of terms, we may repeat the last term infinitely many times. Hence we lose no generality in assuming this sequence to be infinite. Let A_i^0 be the arc of C from a_{i-1} to a_i in the counterclockwise direction. For each i greater than one let A_i^1, A_i^2, A_i^3, $\cdots$, $A_i^{r_i-1}$ be r_i-1 independent arcs (numbered inwardly) spanning* C between the points a_{i-1} and a_i. Let these arcs all lie in the sector of the circle C determined by the rays joining the origin to a_{i-1} and a_i and the portion of the circle swept out by the radius vector as it moves from a_{i-1} to a_i with increasing θ. Construct these arcs in such a way that the length of A_i^j is less than twice the length of A_i^0 for all $j \leq r_i-1$. Also arrange that no radius of C meets any A_i^j in more than one point. We may then define A_i^n for all positive integers n by reducing the superscript n, modulo r_i. We define the limit set L by the equation

$$L = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} A_i^n.$$

For every positive integer i define

$$n_i = \sum_{j=1}^{i} 2^{i-j} r_j.$$

We shall now construct an infinite sequence of point sets P_i having the following properties: (i) P_i consists of exactly n_i points, for each i; (ii) P_i lies wholly in the plane S_i: $z=1/i$, $w=0$; (iii) P_i converges to the set L in the plane $z=0$, $w=0$.

Let R_i, $(i=0, 1, 2, \cdots)$, denote the ray joining the origin to the point a_i, for each i. For i fixed consider any two rays R_{j-1} and R_j, $(j \leq i)$. Between these rays, but not including them, construct 2^{i-j} rays equally spaced in the sector and let them be named K_j^1, K_j^2, K_j^3, $\cdots$, $K_j^{2^{i-j}}$ in the counterclockwise order in which they occur. The ray K_j^n will intersect L in exactly r_j points, one on each of the arcs A_j^s, $(s=0, 1, 2, \cdots, r_j-1)$. We arrange all these points of intersection, for $j=1, 2, \cdots, i$, in a linear array $w_1, w_2, \cdots, w_{n_i}$, where w_t is the point of intersection of the arc A_j^n and the ray K_j^s for the value

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* An arc axb is said to span a point set M provided $(axb)M = a+b$.

$$t = n + 1 + (s - 1)r_j + \sum_{m=1}^{j-1} 2^{i-m}r_m.$$

It is easily seen that this orders the points first, according to which sector of the circle they are in, second, according to which ray they are on, and third, according to the free arc of L which contains them. Let w_t have the coordinates $(x_t, y_t, 0, 0)$. Let p_i^t be the point with coordinates $(x_t, y_t, 1/i, 0)$, and define

$$P_i = \sum_{t=1}^{n_i} p_i^t .$$

It follows easily that the sequence of point sets thus constructed satisfies (i), (ii), and (iii).

We have now reached the final stage of the construction which will be made by adding to L an infinite sequence of 2-cells, where each 2-cell is considered as the homeomorph of a triangle and its interior. It is thus meaningful to speak of the sides and vertices of a 2-cell. Every 2-cell will have one of its sides in L; this side will be called the *base* of the 2-cell. The other vertex of the 2-cell will lie in one of the planes S_i and will be referred to as the *apex* of the 2-cell. By the *altitude* we mean the maximum distance from the base to any point of the 2-cell.

For every integer i we shall construct a set of n_i 2-cells G_i^1, G_i^2, $G_i^3, \cdots, G_i^{n_i}$. The bases of the G_i^j will be taken as the sets

$$B_i^j = \sum_{k=1}^{i} A_k^j .$$

The apexes b_i^j will be chosen to satisfy the following conditions: (a) all the b_i^j are distinct; (b) for each (i, j), $G_i^j S_i = b_i^j$; (c) $\rho(b_i^j, p_i^j)$ $<1/i$. We finally require that for every $\epsilon>0$ there exist at most a finite number of G_i^j of altitude greater than ϵ, and that the product of any of the 2-cells, which are not identical, lies in L. All of these conditions are easily satisfied in our four-dimensional space.

Our space M is now defined to be

$$M = L + \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} G_i^j.$$

It is easily seen that M is a locally connected continuum.

We must now define the transformation $T(M) = M$. To do this we define $T(A_i^j) = A_i^{j+1}$ as a homeomorphism having the end points of A_i^j as fixed points. This defines T on L. For each G_i^j let $T(G_i^j) = G_i^{j+1}$

(where the superscript is reduced, modulo n_i, if necessary) be a homeomorphism agreeing with T on L and sending b_i^j into b_i^{j+1} (with the same convention on the superscripts). This defines T for every p of M. It is evident that $T(M) = M$ is a pointwise periodic homeomorphism.

If we now define

$$G_i = \sum_{j=1}^{n_i} b_i^j,$$

we see that each G_i is an orbit under T, and conditions (a) and (b) of the theorem are satisfied. The proof is thus complete.

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AN ENUMERATION OF LOGICAL FUNCTIONS

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In a logical calculus of m values, abbreviated by L_m, we may deal with functions of n variables. A particular function is defined in this calculus if we assign a constant value, which may be any arbitrary one of the m possible values in L_m, as the value of that function for a particular argument. It is the purpose of this note to enumerate, among all functions of n variables in L_m: those which depend on all n variables in the argument; those which depend on just $(n-1)$ of the variables in the argument, being independent of one of them; and so on; finally those which are completely independent of all the variables in the argument.

Since each variable in the argument may assume values from $1, \cdots, m$, independently, there are m^n possible arguments, and since to each argument we may assign independently, as a functional value, any of the m values $1, \cdots, m$, there are in all m^{m^n} possible functions of n variables.

Let V_n be the total number of all functions of n variables in L_m. Then we have from the above

(1) $$V_n = m^{m^n}.$$

Let U_{nk} be the number of functions of n variables which depend on exactly k of them. (It is this expression for which we are seeking an explicit evaluation.) Since k variables may be selected from n of them in just $C_{n,k}$ ways, we have the relation:

(2) $$U_{nk} = C_{n,k} U_{kk}.$$