SHORTER NOTICES

Die Gegenwärtige Lage in der mathematischen Grundlagenforschung. Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie. By Gerhard Gentzen. (Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, new series, no. 4.) Leipzig, Hirzel, 1938. 44 pp.

This is really two books printed in one volume. The first one, "Die gegenwärtige Lage in der mathematischen Grundlagenforschung," has also appeared in Deutsche Mathematik, vol. 3 (1938), pp. 255–268. The second one, "Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie," is a revision of Gentzen's famous paper in the Mathematische Annalen, vol. 112 (1936), pp. 493–565.

The first book is a well written summary of the present status of foundations, and contains one of the most lucid accounts of the Brouwer viewpoint that the present reviewer has seen. The distinction between the Brouwer and Hilbert schools is presented from the point of view of their treatment of the infinite. For Brouwer, who always insists on finite constructibility, the infinite exists only in the sense that he can at any time take a larger (finite) set than any which he has taken hitherto. Hilbert would treat of infinite sets by the same methods used for finite sets, as if he could comprehend them in their entirety. Gentzen refers to this point of view as the "as if" point of view. He presents various paradoxes which arise when the "as if" method is used without proper care.

This of course opens the question of what is "proper care." In the nature of things, the Brouwer method must fail to produce a paradox, since it never leaves the domain of the constructive finite. However the Brouwer method does not produce sufficient mathematical theory for physical and engineering uses. So Brouwer's method must be described as "excessive care."

A proposed way out of the difficulty is to base the "as if" method on an appropriate formal system, and use the Brouwer method to prove that the formal system is without a contradiction. For none of the various formal systems so far proposed has such a proof of freedom from contradiction been given. More serious still, a well known theorem of Gödel says that if a logic L_1 is used to prove the freedom from contradiction of a logic L_2, then L_1 must in some respects be stronger than L_2. So the above program will fall through unless one can point out some respect in which the Brouwer method is stronger than the "as if" method. Gentzen thinks he has found it.

His idea is to use the Brouwer method, involving the use of transfinite induction up to a certain ordinal α, to prove the freedom from contradiction of that part of the "as if" method which involves transfinite induction only up to an appropriate smaller ordinal β. If β is fairly large, the resulting "as if" method, though restricted, should be adequate for physics and engineering.

In the second book, Gentzen illustrates the above proposal by using the Brouwer method, with induction up to ϵ_0, to prove the freedom from contradiction of number theory with induction up to any ordinal less than ϵ_0. An important gap in the proof is the absence of a *constructive* proof that induction is valid up to ϵ_0. Gentzen himself comments on this gap, and expresses the belief that it will shortly be filled.

The present proof of freedom from contradiction is made considerably simpler than the earlier proof (in Mathematische Annalen—see above) by using Gentzen's LK-calculus, rather than his NK-calculus.

The proof is too complicated to be sketched here. However it is worth saying that what Gentzen does is to describe a means of attaching an ordinal number (less than ϵ_0) to any proof of number theory. He then describes how, if one had a proof of a contradiction, one could find a second proof of a contradiction having a smaller ordinal number than the first proof.

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Reports of a Mathematical Colloquium. Series 2, no. 1. Edited by Karl Menger. Notre Dame University Press, 1939. 64 pp.

This booklet of seven papers begins a continuation of the earlier series of reports issued from 1928 to 1936 by the Vienna Colloquium under the leadership of Professor Menger.

"Stability of Limited Competition and Cooperation," by G. C. Evans and Kenneth May, deals with two producers. Under simplifying assumptions, conditions are found on the coefficients of the demand and cost functions in order that an equilibrium point be possible, that this point be competitive or cooperative, and that the equilibrium be stable. Under strong hypotheses, similar methods are applied to labor, leading to the conclusion that a union able to control the labor supply for a given industry can make the introduction of machinery unprofitable to the entrepreneur.

"On Linear Sets in Metric Spaces," by Karl Menger and Arthur Milgram, contains four theorems, of which a special consequence is the known theorem that, in a complete and convex metric space, any two distinct points are joined by a subset congruent to a segment of the euclidean line.

The third paper is "Partially Ordered Sets, Separating Systems and Inductiveness," by A. N. Milgram. It is unfortunate that this interesting and substantial paper gives the impression of having been written hurriedly and printed without proofreading. The results given seem to be new and significant.

Essentially, certain portions of the Dedekind theory of the continuum are so adjusted as to be useful in studying partially ordered sets. Let A be a partially ordered set. A subset L is called a lower section of A if $a \varepsilon L$ and $b<a$ imply $b \varepsilon L$; if B is a subset of A, the set of those elements a of A with the property that $a \leqq b$ for every b in B is called the under section of B. In terms of these concepts, it is found possible to define well-ordered subsets and to associate with each element of A a unique subset which is well-ordered and has other useful properties; this association is produced once with and once without an application of transfinite induction. Separation—analogous to that effected in the continuum by the rational numbers—is defined intrinsically and extrinsically, the equivalence of the definitions is proved, and the powers of well-ordered subsets are compared with the powers of systems of separating sets. If A has a denumerable separating system, it is shown that A can be mapped on an interval of the continuum in such a way that order-relations are preserved. Several applications are given, chiefly to problems in topology.

"Postulates for the Ratio of Division," by B. J. Topel, deals with a set of elements and a real-valued function f of trios of these elements. f is subject to postulates which arise in a natural way from the properties of the ratio in which a point divides a segment in elementary geometry. It is shown that the set can be so metrized that f is the quotient of two distances. Limiting processes, orientation, and f-preserving transformations are studied, and other sets of postulates are considered.

Frederick P. Jenks proposes a set of postulates for Bolyai-Lobachevsky geometry based on the operations of joining and intersecting, and shows that these postulates are sufficient for the usual discussion of betweenness.