From (7.15), (7.17), (7.18), we conclude that these ∞^1 V_{m-1} contain axial points. Hence the theorem:

THEOREM. *The V_m in S_n of type* (3) *are either V_m in S_{m+2} or contain ∞^1 V_{m-1} with axial points.*

If the rank of $h_{\lambda\mu}$ (index $n-1$) is greater than two, then the rank of $\overline{B}^{\alpha\beta}_{\lambda\mu}h_{\alpha\beta}$ (h of index $n-1$) is greater than one. In this case, from Segre's theorem, the ∞^1 V_{m-1} lie in ∞^1 S_m. If the rank of $h_{\lambda\mu}$ (index $n-1$) is two and its nonzero *domain** *does not* contain the nonzero domain of $h_{\lambda\mu}$ (index n), then the same result is valid; if it does contain the nonzero domain of this $h_{\lambda\mu}$, then $\overline{B}^{\alpha\beta}_{\lambda\mu}h_{\alpha\beta}$ (index $n-1$) is of rank one. In this last case the ∞^1 V_{m-1} are ∞^1 developable S_{m-1}.

From §§6 and 7, we have the theorem:

THEOREM. *If the rank of any of the two second fundamental tensors is greater than two, then V_m in S_n with planar points are* (1) *V_m consisting of ∞^1 V_{m-1} imbedded in ∞^1 S_{m+1}, or* (2) *V_m consisting of ∞^1 V_{m-1} imbedded in ∞^1 S_m, or* (3) *V_m lying in S_{m+2}.*

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ON TRANSITIVE GROUPS THAT CONTAIN CERTAIN TRANSITIVE SUBGROUPS†

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If a simply transitive permutation group G of compound degree n contains a regular abelian subgroup H of order n, and if at least one Sylow subgroup of H is cyclic, G is imprimitive. The proof of this important theorem, due to Wielandt,‡ is remarkable for its brevity. But familiarity with certain preliminary theorems of Schur's§ is assumed. Unfortunately these theorems, as presented by Schur, do not appear to be as elementary as they really are. It seems, therefore, worth while to offer a complete proof of Wielandt's theorem that is elementary throughout, free from the theories of rings and representations, and based on the fundamental concept of the double coset, introduced by Cauchy|| in 1846. Some generalizations, too, can readily be made.

* Vol. 1, p. 19; German "Gebiet."

† Presented to the Society, December 29, 1938, under the title *A note on transitive groups with regular subgroups of the same degree.*

‡ H. Wielandt, Mathematische Zeitschrift, vol. 40 (1935), p. 582.

§ I. Schur, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1933, p. 598.

|| A. L. Cauchy, Comptes Rendus, vol. 22 (1846), p. 630.

In §4 of this paper it will be shown how, when a transitive group is given, the set of all linear homogeneous substitutions on the same n variables which are commutative with every permutation of the group can be easily and directly obtained.

1. **Double cosets and primary subsets.** The n letters permuted by the transitive group G of order ng are $a, b_1, \cdots$, and G_a is the subgroup of G that fixes the letter a. G_a is to be regarded as an intransitive group of degree n with r transitive constituents, counting those on one letter. If $ng>n$, and $r=2$, G is doubly transitive.

If s is a permutation of G, the double coset G_asG_a is a collection* of g^2 elements, each occurring g/m times if m is the number of letters in the transitive constituent of G_a to which the letter x belongs in case s replaces a by x. If s is the identical permutation e, $G_aeG_a=G_aG_a$, which can be written G_a^2 or gG_a, a collection in which each element of the set G_a is repeated g times. The inverse double coset $(G_asG_a)^{-1}=G_as^{-1}G_a$ has the same number mg of distinct permutations as G_asG_a, so that, if $s=(yax\cdots)\cdots$, y is one of the letters of a transitive constituent of G_a, also of degree m. This is the "pairing" of transitive constituents discovered by Burnside.† The double coset G_asG_a is unchanged if s is replaced by any other permutation of G_asG_a.

Let the crosscut of two finite collections K and L be indicated by the symbol $K\cap L$, and be defined by saying that if an element is repeated k times in K and l times in L, it occurs $(k+l-|k-l|)/2$ times in $K\cap L$.

Let H be a transitive subgroup of degree n and order nh of the transitive group G. The subgroup of H that fixes a is H_a, and is of order h. The crosscut $H\cap(G_asG_a)$ will, if s is a permutation of G, be called a primary subset of H with respect to G_a. The subset H_a of H is a primary subset of H with respect to G_a, even when $h=1$. If $s=s_1=(ab_1\cdots)\cdots$, where b_1 is one of the m letters $b_1, b_2, \cdots, b_m$, of a transitive constituent of G_a, this crosscut consists of mh distinct permutations of H:

$$K:\qquad s_1=(ab_1\cdots)\cdots,\ s_2=(ab_2\cdots)\cdots,\ \cdots,$$
$$s_m=(ab_m\cdots)\cdots;\ \cdots.$$

If a collection consists of these mh permutations of a primary subset,

* A collection becomes a set if we disregard the repetitions of an element. Cf. Kestleman, *Theories of Integration*, 1938.

† W. Burnside, Proceedings of the London Mathematical Society, vol. 33 (1901), p. 162.

each repeated n_1 times, it is written n_1K. A chief collection of H with respect to G_a is a collection of primary subsets of H with respect to G_a, and if $K_1, K_2, \cdots$ are the primary subsets involved in the chief collection K, we may use the notation

$$K = n_1K_1 + n_2K_2 + \cdots + n_qK_q,$$

where $n_1, n_2, \cdots$ are natural numbers. A primary subset as here defined is, when H is regular, equivalent to the "primären Komplex" of Schur, Wielandt and Kochendörffer, but as there is confusion in the use of the word "Hauptkomplex," I prefer to avoid the word "complex" altogether.

It results immediately from the definition that H has exactly r primary subsets, with respect to G_a corresponding to the r transitive constituents of G_a, and any given permutation of H is in one and only one primary subset (with respect to G_a). If K is a primary subset of H with respect to G_a, it is a primary subset of H with respect to every subgroup G_x that fixes one letter x of G if K is invariant under H.* In particular, the phrase "with respect to G_a" can be omitted when H is abelian. Two collections K and L of permutations of G are said to be permutable if $KL=LK$, exact account being taken of the number of times a permutation recurs. With this convention as to the product of two collections we can state a well known theorem as follows:

Two permutation groups U and V, of orders u and v respectively, with w permutations in common, generate a group of order uv/w if and only if $UV=VU$. Then H and G_a are permutable groups.

2. **Criteria for primitivity.** Throughout this section we shall retain the preceding notation: H is any transitive subgroup of degree n and

* The necessary and sufficient condition that a primary subset K of H with respect to G_a be a primary subset of H with respect to every subgroup G_x is that all the transforms of K under H are also primary subsets of H with respect to G_a.

This theorem can be proved as follows. Let $K=H\cap G_xAG_x$ for every letter x of G. Let $t=(xa\cdots)\cdots$ be a permutation of H. Then $t^{-1}Kt=H\cap G_at^{-1}AtG_a$, because $t^{-1}G_xt=G_a$. Note that for $(xa\cdots)\cdots$ we can use any one of the permutations tH_a, because the permutations of H_a transform every primary subset of H with respect to G_a into itself, and $t^{-1}Kt$ is seen to be a primary subset of H with respect to G_a by definition. Now if $K=H\cap G_xAG_x$ for every x, the permutations of a certain number of cosets $tH_a, \cdots$ transform K into K, thus forming a subgroup N of H, and the permutations $t_1N, t_2N, \cdots$ transform K into the primary subsets $K_1, K_2, \cdots$ with respect to G_a. Conversely, if all the transforms of K under H are primary subsets of H with respect to G_a we take n permutations $t=(yax\cdots)\cdots$ from H, y running through the n letters. Now $t^{-1}Kt=H\cap G_xt^{-1}AtG_x=H\cap G_at^{-1}AtG_a$, the last because $t^{-1}Kt$ is a primary subset of H with respect to G_a, by hypothesis. Hence $K=t(t^{-1}Kt)t^{-1}=H\cap G_yAG_y$, for every letter y of G. June 20, 1939.

order nh of a transitive group G of degree n and order ng: H_a and G_a are the subgroups fixing one letter a of H and G respectively. Our fundamental theorem is the following:

THEOREM 1. *A collection K of permutations of H is a chief collection of H with respect to G if and only if $H_aK = hK$ and $G_aK = KG_a$.*

First, let K be a primary subset of H with respect to G_a composed of the mh permutations of $H_as_1, H_as_2, \cdots, H_as_m$, where

$$s_1 = (c_1ab_1 \cdots) \cdots, s_2 = (ab_2 \cdots) \cdots, \cdots,$$
$$s_m = (ab_m \cdots) \cdots.$$

Evidently $H_aK = hK$. The set of mg permutations in the m cosets $G_as_1, G_as_2, \cdots, G_as_m$ consists of the distinct permutations of the double coset $G_as_1G_a$; therefore, exactly the same permutations occur in $s_1G_a, s_2G_a, \cdots, s_mG_a$. Thus $G_aK = KG_a$, when K is a primary subset. If $K = H_a$, $H_aK = hK$ and $G_aK = KG_a$. It follows that, if K is a chief collection of H with respect to G_a, $H_aK = hK$ and the collections G_a and K are permutable.

To prove the converse, let K be a collection of permutations of H which satisfies the two conditions of the theorem. Then $G_aG_aK = G_aKG_a$, and we expand this by putting for K each of its permutations (call them $s_1, s_2, \cdots$), each as often as it occurs in K, so that

$$gG_aK = n_1(G_as_1G_a) + n_2(G_as_2G_a) + \cdots.$$

The only permutations of H in G_aK are the permutations of H_aK, which by hypothesis is hK; therefore

$$ghK = n_1(H \cap G_as_1G_a) + n_2(H \cap G_as_2G_a) + \cdots,$$

that is, ghK is a chief collection. Therefore K is a chief collection with respect to G. For, if K is not a chief collection, ghK is not one.

THEOREM 2. *If K and L are two chief collections of H with respect to G_a, KL is a chief collection with respect to G_a.*

For $HKL = hKL$ and $G_aKL = KG_aL = KLG_a$.

A chief collection of H is said to be singular if its permutations generate a proper subgroup of H of order greater than h.

THEOREM 3. *The group G is imprimitive if and only if some chief collection of H is singular.*

Let G be imprimitive. Then G_a is a proper subgroup of a proper subgroup J of G. This J does not include H, but it does contain at least one permutation of the set $H-H_a$. Let s be a permutation of $J-G_a$. Since every element of G_asG_a is in J, $H \cap G_asG_a$ is a subset of $H \cap J$, which last is a proper subgroup of H different from H_a; hence $H \cap G_asG_a$ is a singular primary subset.

Let K be a singular chief collection of H with respect to G_a. Its elements generate a proper subgroup C of H of order greater than h. Since C is of finite order, there exists a natural number N such that every permutation of C is in the chief collection $K+K^2+\cdots+K^N$; of course every element of this collection is in C, so that C is a chief collection of H with respect to G_a. By Theorem 1, H_a is a subgroup of C, and $CG_a=G_aC$, that is, C and G_a are permutable groups with H_a in common, and therefore CG_a is a proper subgroup of G, which means that G is imprimitive.

The following theorem is an extension of Wielandt's lemma:

THEOREM 4. *Let P be a proper subgroup of H but not a subgroup of H_a. If the elements of a set of left (or right) cosets of H with respect to P constitute a chief collection K of H with respect to G_a, and if $K \neq H$, G is imprimitive.*

If P alone, or if P and certain cosets with respect to P, constitute a chief collection L of H ($L\neq H$), then $K=H-L$ is a chief collection composed of cosets with respect to P. Hence it is sufficient to discuss the case in which K does not contain P. Then there are at least two primary subsets involving permutations of P (e is in one of them) and at least one in K, so that H has at least three primary subsets with respect to G_a, which means that G_a has at least three transitive constituents. Let $H_as_1, H_as_2, \cdots, H_as_k$, where $s_1=(ay_1\cdots)\cdots$, $s_2=(ay_2\cdots)\cdots, \cdots, s_k=(ay_k\cdots)\cdots$, be the permutations of the collection K. Now $H_as_1P, H_as_2P, \cdots, H_as_kP$ include (with repetitions) all the left cosets that make up K. Let $t=(y_iz\cdots)\cdots$ be a permutation of P. Since the product $s_it=(az\cdots)\cdots$ belongs to K, z is one of the letters $y_1, y_2, \cdots, y_k$, and therefore P permutes the letters of P only among themselves. But G_a also permutes these letters $y_1, y_2, \cdots, y_k$ among themselves and in consequence the group generated by P and G_a is intransitive. Hence G_a is a proper subgroup of a proper subgroup of G, and G is imprimitive.

The following theorem is due to Schur, and the proof which we shall briefly indicate is his.*

THEOREM 5. *Let G be a primitive group and let the transitive group H be abelian. If p is a prime divisor of the degree n, and if r is any permuta-*

* I. Schur, loc. cit.

tion of H other than the identity, the number of solutions of $x^p = r$ in a primary subset K of H is congruent to zero, modulo p.

Let $s_1, s_2, \cdots, s_m$ be the permutations of the primary subset K. In the chief collection K^p the number of times any element, except perhaps $s_1^p, s_2^p, \cdots, s_m^p$, occurs is, because H is abelian, a multiple of p. If from K^p we remove those primary subsets which occur kp times ($k=1, 2, \cdots$), the remaining collection, if not empty, will be a chief collection of H consisting of some or all of the permutations $s_1^p, s_2^p, \cdots, s_m^p$. Now $s_1^p, s_2^p, \cdots, s_m^p$ generate a proper subgroup of H, and therefore the chief collection remaining generates a proper subgroup J of H, which is not allowable unless $J=e$ because, by hypothesis, G is primitive. Hence after the removal of certain permutations because they occur in sets of p, only e, the identity, can remain. Thus $x^p = r$, where $r \neq e$, either has no solution in K or kp of the permutations $s_1, s_2, \cdots, s_m$ of K satisfy it.

3. **Wielandt's theorem.** We are now prepared to prove

THEOREM 6. *If a simply transitive group G not of prime degree contains a regular abelian subgroup H of the same degree, and if one of the Sylow subgroups of H is cyclic, G is imprimitive.*

Let K be a primary subset (not e) of H, and let p be the prime to which there corresponds a cyclic Sylow subgroup of H. If x_0 is an element of K such that $x_0^p = r$, and if $x_0 y$ is another solution in K, $y^p = e$. But because H has only one subgroup of order p, only p permutations of H satisfy $y^p = e$. If $P = \{s\}$ is this cyclic subgroup of order p, $x_0, x_0 s, \cdots, x_0 s^{p-1}$ are the only possible solutions of $x^p = r$ in K. Therefore, if K contains no permutation of order p, and if G is primitive, all the permutations of K lie in the cosets $x_0 P, x_1 P, \cdots$, so that by Theorem 4 the group G should be imprimitive. Since H has at least three primary subsets, we can, when $p=2$, choose K so that $K \cap P$ is empty. In the following, then, p is an odd prime. If K is a subset of P, K is singular because of the condition that p is less than n; this again would make G imprimitive.

If G is primitive, $K = XP + F$, where the set X consists of permutations $x_0, x_1, \cdots$ of K whose pth powers are in $H - e$, and no two of which have the same pth power; and the set F consists of the f distinct permutations $s^{n_1}, s^{n_2}, \cdots, s^{n_f}$ of order p. Neither X nor F is empty. Now $P^2 = pP$, $PF = fP$, and in F^2 no permutation can occur more than f times. Hence (H being abelian) $K^2 = pX^2P + 2fXP + F^2$.

Let us say that we have chosen for our primary subset K a primary subset of H for which $f \leqq (p-1)/2$. Since K^2 is a chief collection and

$2fXP$ contains a permutation of K repeated at least $2f$ times, K^2 contains $2fK=2fXP+2fF$. Then K^2-2fK is a chief collection and is equal to $(pX^2P+F^2)-2fF$. No element of F^2 occurs more than f times, so that every element of F is an element of X^2P. If then from $(pX^2P+F^2)-2fF$ we remove primary subsets p at a time until no permutation left occurs more than $p-1$ times, we get a chief collection $F^2+(p-2f)F$, clearly singular, implying that G is imprimitive.

Thus Wielandt's theorem is proved, and the proof is Wielandt's, stripped to show its truly elementary character.

4. **Linear substitutions.** Let G be a transitive group on the n letters $x_1, x_2, \cdots, x_n$. Let G_k be its subgroup that fixes the one letter x_k. We agree as before that G_k is to be regarded as a group of degree n with r transitive constituents. Let one of the transitive constituents of G_1 be on the m letters (or variables if you like) $x_{11}, x_{12}, \cdots, x_{1m}$, where x_{1j} is of course one of the letters $x_1, x_2, \cdots, x_n$. Let t_k be a permutation of G that replaces the $m+1$ letters $x_1, x_{11}, x_{12}, \cdots, x_{1m}$ by $x_k, x_{k1}, x_{k2}, \cdots, x_{km}$, respectively. Then $t_k^{-1}G_1t_k=G_k$, and in G_k is found a transitive constituent on $x_{k1}, x_{k2}, \cdots, x_{km}$. Thus there is a one-to-one correspondence between the n subgroups $G_1, G_2, \cdots, G_n$ and the n sets of letters $x_{1j}, x_{2j}, \cdots, x_{nj}$ $(j=1, 2, \cdots, m)$. Then the linear substitution

$$\begin{aligned} x_1' &= x_{11} + x_{12} + \cdots + x_{1m}, \\ x_2' &= x_{21} + x_{22} + \cdots + x_{2m}, \\ &\cdots\cdots\cdots\cdots\cdots, \\ x_n' &= x_{n1} + x_{n2} + \cdots + x_{nm} \end{aligned} \tag{1}$$

is transformed into itself by every permutation of G. Neither the n subgroups G_k nor the n sets of m letters need be distinct. In this way r substitutions can be set up, and if each of them is multiplied by an arbitrary constant and they are then added, the result is a substitution commutative with every permutation of G. If $m=1$, (1) is a permutation. If G is a regular group, $r=n$, and we get n distinct permutations, that is, the conjoin of G.

Let us suppose now that the substitution

$$V: \qquad x_i' = \sum_{j=1}^{n} v_{ij}x_j, \qquad i = 1, 2, \cdots, n,$$

is commutative with every permutation of G. Consider the first equation $x_1' = \sum_{j=1}^{n} v_{1j}x_j$ of V. When V is transformed by all the permutations of G_1, x_1' is invariant and therefore the m coefficients of the variables $x_{11}, x_{12}, \cdots, x_{1m}$ must be equal, to v_1, say. Next, consider the transform of V by t_k (defined in the preceding paragraph): x_1' goes into x_k', and the sum $v_1(x_{11}+x_{12}+\cdots+x_{1m})$ goes into $v_1(x_{k1}+x_{k2}+\cdots+x_{km})$, so that the variables $x_{k1}, x_{k2}, \cdots, x_{km}$ in $x_k' = \sum_{j=1}^{n} v_{kj}x_j$ all have the same coefficient v_1, for $k=1, 2, \cdots, n$. Hence:

THEOREM 7. *Every linear substitution commutative with each of the permutations of a transitive group can be obtained by first setting up the substitutions* (1) *and then multiplying them by appropriate constants and adding.*

If the given transitive group G contains a regular subgroup H of the same degree n, the permutations $t_1=e, t_2, t_3, \cdots, t_n$ can be the permutations of H, and the substitution (1) will be the sum of m permutations of the conjoin of H. There is a connection between the substitution (1) and a primary subset of H.

THEOREM 8. *Let G be a transitive group of degree n and H a regular subgroup of order n. Let the primary subset K of H be transformed into itself by every permutation of H. Then the sum of the permutations of K is commutative with every permutation of G.*

This K is independent of our choice of subgroup fixing one letter. We return to the notations of §1. If $H, H_1, \cdots, H_{w-1}$ is the complete set of conjugates to which H belongs under G, it is also, because $HG_x=G$, a complete set of conjugates under G_x. Those permutations of G_x which transform H into H, transform K into K. Hence K has one and only one primary subset conjugate to it in each of the subgroups $H, H_1, \cdots, H_{w-1}$. These w primary subsets $K, K_1, \cdots, K_{w-1}$, whether distinct or not, constitute a conjugate set under G. If in the permutations of K the letter x is followed by $y_1, y_2, \cdots, y_m$, it is followed by the same m letters in some order in K_1, in $K_2, \cdots$, in K_{w-1}. Therefore the sum of the permutations of K_i is equal to the sum of the permutations of K $(i=1, 2, \cdots, w-1)$.

This result applies to every primary subset of H when H is abelian. The following theorem is more general.

THEOREM 9. *Let G be a transitive group of degree n and H a regular subgroup of order n. Let G_a be the subgroup of G that fixes the one letter a, and let K be a primary subset of H with respect to G_a, composed of the m permutations*

$$t_1 = (c_1ab_1 \cdots) \cdots, t_2 = (c_2ab_2 \cdots) \cdots, \cdots,$$
$$t_m = (c_mab_m \cdots) \cdots .$$

Let K^* be the subset of the conjoin of H composed of the m permutations

$$t_1^* = (ac_1 \cdots) \cdots, \quad t_2^* = (ac_2 \cdots) \cdots, \quad \cdots, \quad t_m^* = (ac_m \cdots) \cdots .$$

Then the sum of the permutations of K^* is a linear substitution commutative with every permutation of G.

There exists a permutation $(a)(b_1c_1)(b_2c_2) \cdots (b_mc_m) \cdots$ of order 2 which transforms the regular group H into its conjoin H^*.

Any permutation s that transforms H into H_i, transforms H^* into the conjoin H_i^* of H_i; because if every permutation of H^* is commutative with every permutation of H, then every permutation of $s^{-1}H^*s$ is commutative with every permutation of $s^{-1}Hs$. Let N_a be the cross-cut of G_a and N, the normalizer of H in G. The permutations of N_a transform each primary subset K of H with respect to G_a into itself. The w conjugate subgroups $H, H_1, \cdots, H_{w-1}$ contain w primary subsets with respect to G_a: $K, K_1, \cdots, K_{w-1}$, forming, though perhaps not all distinct, a conjugate set under G_a, as do also the corresponding subsets $K^*, K_1^*, \cdots, K_{w-1}^*$ of $H^*, H_1^*, \cdots, H_{w-1}^*$ (respectively). The permutation $t=(ax \cdots) \cdots$ of H transforms G_a into G_x, and therefore replaces the letters $c_1, c_2, \cdots, c_m$ of a transitive constituent of G_a by the letters $z_1, z_2, \cdots, z_m$ of a transitive constituent of G_x. But t transforms each permutation of K^* into itself. Hence in K^*, x is followed by $z_1, z_2, \cdots, z_m$. Now

$$G_a = N_a + N_aq_1 + N_aq_2 + \cdots + N_aq_{w-1},$$

where each permutation of N_aq_i transforms K^* into K_i^*. The permutation q_i is in HG_x, and every permutation of H is commutative with every permutation of K^*, while every permutation of G_x permutes among themselves the m letters $z_1, z_2, \cdots, z_m$ that follow x in the permutations of K^*. This being true for every letter x of G, and for every i, the sum of the permutations of K_i^* is equal to the sum of the permutations of K^*.

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