CONCERNING CERTAIN LINEAR ABSTRACT SPACES AND SIMPLE CONTINUOUS CURVES*

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The first section of this paper is given over mainly to the investigation of linear Hausdorff spaces. However, the principal object of the paper is to characterize topologically that class of point sets used in the geometry for lines, namely, the class of simple continuous curves. This has already been done by R. L. Moore,† by R. L. Wilder,‡ and by myself.§ The results of this paper generalize the results just referred to mainly by omitting all compactness requirements. As a matter of fact, it will be shown that any nondegenerate linear continuum lying in a Moore space is a simple continuous curve, and that any nondegenerate linear connected subset of a Moore space is homeomorphic with a simple continuous curve.

1. Certain results for Hausdorff spaces.

DEFINITION. *A space is said to be strongly regular at a point P provided that, if R is a region containing P, then there exists in R a domain D containing P whose boundary is a subset of the sum of a finite number of continua lying in $R-D$.*|| *A space is said to be strongly regular provided that it is strongly regular at every one of its points.*

THEOREM 1. *If P is a point of a connected Hausdorff space M and M is strongly regular at P, then M is connected im kleinen at P.*

PROOF. Let R denote a region containing P, and let R_1 denote a region containing P which lies together with its boundary in R. There exists in R_1 a domain D containing P whose boundary is a subset of the sum of a finite number of continua $T_1, T_2, \cdots, T_n$ lying in

* Presented to the Society, December 30, 1938, under the title *Concerning simple linear Moore spaces and simple continuous curves.*

† R. L. Moore, *Concerning simple continuous curves*, Transactions of this Society, vol. 21 (1920), pp. 313–320. Also R. L. Moore's *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932; Theorem 20′ of chapter 2 in particular. Hereinafter, this book will be referred to as *Foundations.*

‡ R. L. Wilder, *Concerning simple continuous curves and related point sets*, American Journal of Mathematics, vol. 53 (1931), pp. 39–55.

§ F. B. Jones, *Concerning the boundary of a complementary domain of a continuous curve*, this Bulletin, vol. 45 (1939), pp. 428–435.

|| See Axiom 5^*_1 of my paper *Concerning certain topologically flat spaces*, Transactions of this Society, vol. 42 (1937), pp. 53–93.

R_1-D. Since M is connected, $D+T_1+T_2+\cdots+T_n$ is the sum of at most n components. Hence, if C denotes the component containing P, $C\cdot D$ is an open set. But, since C lies in R, M is connected im kleinen at P.

THEOREM 2. *A strongly regular connected Hausdorff space is locally connected.*

DEFINITION. *A point set M is linear provided that, if P is a point of R, a domain with respect to M, there exists in R a domain with respect to M which contains P and has at most two boundary points with respect to M.*†

THEOREM 3. *A linear Hausdorff space is regular.*

PROOF. If P is a point of a region R, there exists in R a domain D which contains P and has at most two boundary points A_1 and A_2. For each i, $(i=1, 2)$, let R_i denote a region containing P, such that $\overline{R}_i$ does not contain A_i. Then there exists a region U containing P and lying in $D\cdot R_1\cdot R_2$. Obviously $\overline{U}$ is a subset of D and, hence, of R.

THEOREM 4. *If P is a point of a region R in a connected linear Hausdorff space, there exists a connected domain D containing P which has at most two boundary points such that $\overline{D}$ is a subset of R.*

PROOF. By Theorem 3, there exists a region U which contains P and lies together with its boundary in R. Let D_1 denote a domain which lies in U and contains P and has at most two boundary points. Since the space is linear and regular, it is strongly regular, and it follows from Theorem 2 that the component D of D_1 which contains P is itself a domain. Obviously, D has at most two boundary points and lies together with these points in R.

THEOREM 5. *If D is a connected domain in a connected linear Hausdorff space, then D has at most two boundary points.*

PROOF. Suppose that D has three boundary points A_1, A_2, and A_3. Let P denote a point of D. Let G denote the collection of all connected domains which have at most two boundary points. There exist three elements d_{11}, d_{21}, and d_{31} of G containing A_1, A_2, and A_3 respectively such that $\overline{d}_{11}\cdot\overline{d}_{21}=\overline{d}_{11}\cdot\overline{d}_{31}=\overline{d}_{21}\cdot\overline{d}_{31}=0$. For each i, $(1\leqq i\leqq 3)$, there exists a chain C_i of elements $d_{i1}, d_{i2}, \cdots, d_{in_i}$ of G from A_i to P such that $\overline{d}_{ij}$ is a subset of D if $1<j\leqq n_i$. For each i, $(1\leqq i\leqq 3)$, let D_i denote the component of $D\cdot C_i^*$ which contains P, and let A_i' de-

† The meaning of the term *linear* as used here should not be confused with its geometric, algebraic, or function-theoretic meaning.

note a boundary point of D_i in the boundary of D.† Obviously, A_i' is in d_{i1}. Now for each i, $(1 \leqq i \leqq 3)$, let C_i' denote a chain of elements $d_{i1}', d_{i2}', \cdots, d_{im_i}'$ of G from A_i' to P such that (1) d_{i1}' contains A_i', (2) $\overline{d}_{i1}'$ is a subset of d_{i1} and contains no point of $\overline{C}_{i+1}^*+\overline{C}_{i+2}^*$ (where $3+1$ in the subscripts is interpreted to mean 1), and (3) if $1<j\leqq m_i$, $\overline{d}_{ij}'$ is a subset of D_i. Obviously, for each i, $(1\leqq i\leqq 3)$, and j, $(1<j<m_i)$, d_{ij}' has one of its boundary points in $d_{i(j-1)}'$ and the other in $d_{i(j+1)}'$. The connected domain $C_1'^*$ has at most two boundary points, of which one is a boundary point of d_{1m_1}' and the other (if it exists) is a boundary point of d_{11}'. Likewise, $C_2'^*$ is a connected domain having at most two boundary points, of which one is a boundary point of d_{2m_2}' and the other (if it exists) is a boundary point of d_{21}'. Since $C_1'^*$ contains no point of $\overline{d}_{21}'$, and $C_2'^*$ contains no point of $\overline{d}_{11}'$, but each contains both a point of and a point not of the other, $C_1'^*+C_2'^*$ is a connected domain which contains P, but not A_3', and whose boundary (if it exists) is a subset of $\overline{d}_{11}'+\overline{d}_{21}'$. But $C_3'^*$ is a connected domain containing P and A_3', but no point of $\overline{d}_{11}'+\overline{d}_{21}'$, which is a contradiction.

THEOREM 6. *A connected linear Hausdorff space is normal.*

PROOF. Suppose that H and K are two mutually exclusive closed subsets of a connected linear Hausdorff space M. Let G denote the collection of all the components of $M-H$ which contain points of K. If C is an element of G, then C is a connected domain and, by Theorem 5, has at most two boundary points A_1 and A_2. Since both A_1 and A_2 are points of H, there exist two regions R_1 and R_2 containing A_1 and A_2, respectively, such that $(\overline{R}_1+\overline{R}_2)\cdot(C\cdot K)=0$. Let C' denote $C-C\cdot(\overline{R}_1+\overline{R}_2)$. Obviously C' is a domain containing $C\cdot K$, such that $\overline{C}'\cdot H=0$. Let G' denote the collection of all such domains C' for all of the elements C of G. Then G'^* contains K and $\overline{G'^*}\cdot H=0$, because if P were a point of H and a limit point of G'^*, then every region with at most two boundary points which contains P would contain points of infinitely many elements of G and, consequently, contain an element of G. Since every element of G contains a point of K, this is a contradiction.

THEOREM 7. *A linear Hausdorff space is atriodic.*

PROOF. Suppose, on the contrary, that the space contains three nondegenerate continua M_1, M_2, and M_3 having only the point P in common. For each i, $(i=1, 2, 3)$, let A_i denote a point of M_i-P.

† The notation C_i^* denotes the sum of the elements of C_i.

There exists a region R containing P and not containing a point of $A_1+A_2+A_3$. Then every domain containing P and lying in R has at least three boundary points, and the space is not linear, contrary to hypothesis.

THEOREM 8. *A connected linear Hausdorff space is locally compact.*

PROOF. Suppose that P is a point of a region R. By Theorem 4, R contains a connected domain containing P, which lies together with its boundary in R and has at most two boundary points. If it exists, let D denote a connected domain containing P such that (1) $\overline{D}$ is a subset of R, and (2) D has two boundary points A and B. Otherwise, let Q denote a connected domain containing P, having only one boundary point B and lying together with its boundary in R, and let D denote $Q-P$. In this last case, D is connected; for, if $Q-P$ contains two components H and K, then, since P is in the boundary of each of them, there exists a connected domain containing P having two boundary points and lying together with its boundary in R. So, in either case, D is a connected domain having two boundary points A and B (in the second case $A=P$). Now suppose that D contains an infinite point set M which has no limit point. If X is any point of M, $M-X$ is not connected. For, if $M-X$ were connected, then $M-X$ would be a connected domain with three boundary points A, B, and X, contrary to Theorem 5. Likewise, since each component of $M-X$ has X on its boundary, no component of $M-X$ has both A and B on its boundary. Furthermore, $M-X$ does not contain three components; for if it did, then, since each such component has X on its boundary, these three components together with X would contain a triod, contrary to Theorem 7. Consequently, for each point X of M, $M-X$ is the sum of two components of $M-X$, AX and BX, having boundaries $A+X$ and $B+X$ respectively. Since M is infinite, it follows that either (1) there exists an infinite sequence $X_1, X_2, X_3, \cdots$ of points of M such that, for each integer n, AX_{n+1} contains AX_n, or (2) there exists an infinite sequence $Y_1, Y_2, Y_3, \cdots$ of points of M such that, for each integer n, BY_{n+1} contains BY_n. Since M has no limit point, $\sum AX_n$ (if the sequence $X_1, X_2, X_3, \cdots$ exists) is a domain having at most one boundary point, A. But $D+B$ is a connected point set which contains the point X_1 of $\sum AX_n$, and which contains the point B of the complement of $\sum AX_n$, but which does not contain A. This is a contradiction. On the other hand, if the sequence $X_1, X_2, X_3, \cdots$ does not exist, then $\sum BY_n$ is a domain, and $D+A$ is a connected point set containing both a point of $\sum BY_n$ and a point not of $\sum BY_n$, but containing no boundary point of $\sum BY_n$, which is again a contradiction. Hence every infinite subset of D has at least one limit point. It follows at once that the space is locally compact.

THEOREM 9. *In a connected linear Hausdorff space every connected domain which has two boundary points is compact.*

Theorem 9 follows from the argument for Theorem 8.

THEOREM 10. *In order that a connected linear Hausdorff space be metric it is necessary and sufficient that it be completely separable.*

PROOF. That the condition is necessary follows from Theorem 8 and a well known result of Alexandroff's.* That the condition is sufficient follows from Theorem 6 and a theorem of Urysohn's.†

In fact, it is rather easy to see that, if S is a connected linear Hausdorff space, the following conditions are equivalent:

(1) *S is separable.*
(2) *Every uncountable subset of S contains a limit point of itself.*
(3) *S has the Lindelöf property.*
(4) *S is completely separable.*
(5) *S is metric.*

2. Applications to Moore spaces.

DEFINITION. *A Moore space is a space satisfying Axiom* 0 *and parts* (1), (2), *and* (3) *of Axiom* 1 *of Foundations. A complete Moore space is one satisfying Axioms* 0 *and* 1 *of Foundations.*

THEOREM 11. *A nondegenerate connected linear Moore space is a simple continuous curve.*‡

PROOF. Since a Moore space is a Hausdorff space, it follows from Theorems 4, 7, and 8 that the space is a locally compact, locally con-

* Paul Alexandroff, *Über die Metrisation der im kleinen kompakten topologischen Räume*, Mathematische Annalen, vol. 92 (1924), pp. 294–301.

† Paul Urysohn, *Zum Metrisationsproblem*, Mathematische Annalen, vol. 94 (1925), pp. 309–315.

‡ Strictly speaking, if one used the definitions of arc, simple closed curve, open curve, and ray as given in *Foundations*, then every nondegenerate connected linear Hausdorff space would be a simple continuous curve. But there would then exist a simple continuous curve in a Hausdorff space which would not be topologically equivalent to one in the number plane. Consequently, in addition to the properties of simple continuous curves set forth in the definitions in *Foundations*, I would require that, in order for a point set (in a Hausdorff space) to be a simple continuous curve, it must also be separable. For *every separable, connected linear Hausdorff space is topologically equivalent to a plane simple continuous curve.*

nected, atriodic complete Moore space. By a theorem of mine,* such a space if nondegenerate is a simple continuous curve.

It can be shown with the help of the theorems and discussion on pages 81, 82, and 83 of *Foundations* and Theorem 11 of this paper that the following three propositions are true.

THEOREM 12. *Every nondegenerate continuum in a linear Moore space M is a simple continuous curve.*

THEOREM 13. *Every nondegenerate linear continuum in a Moore space is a simple continuous curve.*

THEOREM 14. *Every nondegenerate linear connected subset of a Moore space is homeomorphic with a simple continuous curve.*

THEOREM 15. *In order that a nondegenerate, locally connected, complete Moore space be a linear space, it is necessary and sufficient that it contain no simple triod.*

PROOF. If the space is not linear, it is clear that it contains a connected domain with at least three boundary points A, B, and C. Let D_A, D_B, and D_C denote three mutually exclusive connected domains containing A, B, and C respectively. By Theorem 1 in Chapter II of *Foundations*, there exists an arc AB from A to B lying in $D+D_A+D_B$. Likewise, there exists an arc T from C to AB lying in $D+D_C$ and having only one point in common with AB. Obviously, $AB+T$ contains a simple triod. Hence, the condition is sufficient. By Theorem 7, the condition is necessary.

THEOREM 16. *If a nondegenerate continuous curve M in a complete Moore space contains no simple triod, then M is a simple continuous curve.*†

Theorem 16 may be established with the help of Theorems 118 and 120 in Chapter I and the argument for Theorems 6 and 7 in Chapter II of *Foundations* together with Theorems 11 and 15 above.

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* Lemma A of my paper, *Concerning the boundary of a complementary domain of a continuous curve*, loc. cit.

† Theorem 16 is identical with Lemma A of my paper just referred to, except that in Theorem 16 it is not stipulated that M be locally compact.