Using Lemma 3 we have

$$S_n^{(k-1)} = \sum_{j=0}^{k-1} \frac{(-1)^{j+n} C_{n+k,j}}{2^k} \Delta^j a_0 + C_{n+k,k-1} \sum_{j=0}^{k-1} \frac{\Delta^j a_0}{2^{j+1}} + O(n^{k-2})$$

$$= \frac{(-1)^{k+n-1} C_{n+k,k-1}}{2^k} \Delta^{k-1} a_0 + C_{n+k,k-1} \sum_{j=0}^{k-1} \frac{\Delta^j a_0}{2^{j+1}} + O(n^{k-2}).$$

Then

$$\lim_{n\to\infty} c_n^{(k-1)} = \lim_{n\to\infty} \frac{S_n^{(k-1)}}{C_{n+k-1,k-1}} = \frac{\Delta^{k-1} a_0}{2^k} \lim_{n\to\infty} (-1)^{k+n-1} + \sum_{j=0}^{k-1} \frac{\Delta^j a_0}{2^{j+1}}.$$

This limit fails to exist. Consequently, under the hypotheses of our theorem, the series $\sum_{n=0}^{\infty}(-1)^n a_n$ is not summable $(C, k-1)$.

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A GENERAL CONTINUED FRACTION EXPANSION*

WALTER LEIGHTON AND W. T. SCOTT

Introduction. Considerable attention has been given at various times by many writers to the function-theoretic character of continued fractions of the form

$$1 + \frac{a_1 x}{1} + \frac{a_2 x}{1} + \cdots.$$

Only a very restricted class of power series, the "seminormal" ones, admit an expansion into a continued fraction of this type (cf. Perron [3, p. 301]). For example, the power series expansion about the origin of the function $1+x^2$ fails to be seminormal. In §1 of this paper we show that *every* power series admits an expansion into a continued fraction of a form which is a generalization of that above. Many of the older theorems have immediate generalizations. These are presented without proof when the demonstration parallels that for the seminormal case.

In §2 we discuss the question of gaps in seminormal power series. In §3 an important special case is considered.

1. Expansions in continued fractions. Let

$$(1.1) \qquad 1 + \frac{a_1 x^{\alpha_1}}{1} + \frac{a_2 x^{\alpha_2}}{1} + \cdots$$

* Presented to the Society, September 6, 1938.

be a continued fraction, terminating or not, in which the a_i are complex nonzero constants, x is a complex variable, and the α_i are positive integers.* The nth *approximant* $A_n(x)/B_n(x)$ is defined by means of the usual recurrence relations

$$
\begin{aligned}
&A_0 = 1, && B_0 = 1,\\
(1.2)\qquad &A_1 = 1 + a_1x^{\alpha_1}, && B_1 = 1,\\
&A_n = A_{n-1} + a_nx^{\alpha_n}A_{n-2}, && B_n = B_{n-1} + a_nx^{\alpha_n}B_{n-2},\\
& && n = 2, 3, \cdots .
\end{aligned}
$$

The continued fraction (1.1) is said to *converge* for all values of x for which $\lim_{n\to\infty} [A_n(x)/B_n(x)]$ exists, and when convergent this limit is assigned as its value. The difference of two consecutive approximants of (1.1) is given by the formula (Perron [**3**, p. 16])

$$
(1.3)\qquad \frac{A_n(x)}{B_n(x)} - \frac{A_{n-1}(x)}{B_{n-1}(x)} \equiv \frac{(-1)^{n-1}a_1a_2a_3 \cdots a_nx^{s_n}}{B_{n-1}(x)B_n(x)},
$$

where

$$
s_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n.
$$

By equation (1.3) the Taylor development of the rational function $A_{n-1}(x)/B_{n-1}(x)$ about the origin agrees with the development of $A_n(x)/B_n(x)$ up to but not including the term in x^{s_n}. Hence if (1.1) is nonterminating, (1.1) through (1.3) determines uniquely a *corresponding* power series

$$
(1.4)\qquad 1 + c_1x + c_2x^2 + c_3x^3 + \cdots .
$$

If (1.1) terminates, it is either the constant unity or has the form

$$
1 + \frac{a_1x^{\alpha_1}}{1} + \frac{a_2x^{\alpha_2}}{1} + \cdots + \frac{a_nx^{\alpha_n}}{1}.
$$

In the latter case, the corresponding power series is defined to be the Taylor development of $A_n(x)/B_n(x)$ about the origin. One observes that such a development exists since $A_n(x)$ and $B_n(x)$ are polynomials with the value unity at the origin and do not vanish simultaneously. This last fact is obtained from (1.3), which may be written in the form

$$
(1.3')\qquad A_n(x)B_{n-1}(x) - A_{n-1}(x)B_n(x) \equiv (-1)^{n-1}a_1a_2 \cdots a_nx^{s_n}, \qquad n = 2, 3, \cdots .
$$

* Pringsheim seems to be the only writer who refers to the possibility of introducing integers $\alpha_i \geqq 1$ instead of $\alpha_i = 1$ (Pringsheim [**4**, p. 936]). He limits his discussion, however, to the seminormal case.

Thus, corresponding to any continued fraction (1.1) *there always exists a power series of the form* (1.4).

Conversely, let $p_0(x)$ be any power series of the form (1.4). Define a sequence of power series $\{p_n(x)\}$ by the relations

$$(1.5)\qquad p_{n+1}(x) = a_{n+1}x^{\alpha_{n+1}}/(p_n(x) - 1), \qquad n = 0, 1, 2, \cdots,$$

where the α_n are positive integers chosen together with the complex constants a_n in such a way that if $p_n(x)\not\equiv 1$, $p_{n+1}(0)=1$. If no $p_n(x)\equiv 1$, this process yields an infinite sequence of power series $p_0(x)$, $p_1(x)$, $p_2(x)$, $\cdots$. If some $p_n(x)\equiv 1$, the process terminates and yields a finite set of power series $p_0(x)$, $p_1(x)$, $\cdots$, $p_n(x)$. It will be shown later that a necessary and sufficient condition that some $p_n(x)\equiv 1$ is that $p_0(x)$ represent a rational function near the origin. The relations (1.5) clearly determine uniquely a sequence of complex constants $a_1, a_2, \cdots$ and a sequence of positive integers $\alpha_1, \alpha_2, \cdots$. The continued fraction (1.1) formed with these a_i and α_i is said to *correspond* to (1.4). *Hence, corresponding to every power series* (1.4) *there exists a uniquely determined continued fraction of the form* (1.1).

The following result shows that the correspondence between power series and continued fractions is a reciprocal property.

THEOREM 1.1. *If the continued fraction* (1.1) *corresponds to the power series* (1.4), *the power series* (1.4) *corresponds to the continued fraction* (1.1), *and conversely.*

Let $p_0(x)$, the power series (1.4), be given, and let (1.1) be the corresponding continued fraction. It is readily established by induction that

$$(1.6)\qquad p_0(x) \equiv \frac{A_{n-1}(x)p_n(x) + a_nx^{\alpha_n}A_{n-2}(x)}{B_{n-1}(x)p_n(x) + a_nx^{\alpha_n}B_{n-2}(x)}, \qquad n = 1, 2, 3, \cdots,$$

where $A_{-1}(x)\equiv 1$, $B_{-1}(x)\equiv 0$, and $p_n(x)$ is defined by (1.5). To prove the first part of the theorem it will be sufficient to show that $p_0(x)$ agrees with the Taylor development of $A_n(x)/B_n(x)$ about the origin up to but not including the term in $x^{s_{n+1}}$. To this end one observes that, by (1.6) and (1.3),

$$(1.7)\qquad p_0(x) - \frac{A_n(x)}{B_n(x)} \equiv \frac{(-1)^n a_1a_2\cdots a_{n+1}x^{s_{n+1}}}{p_{n+1}(x)B_n(x)[B_n(x)p_{n+1}(x) + a_{n+1}x^{\alpha_{n+1}}B_{n-1}(x)]},$$
$$n = 0, 1, 2, \cdots.$$

The proof of the first part of the theorem is complete.

Conversely, let the continued fraction (1.1) be given, and let $p_0(x)$, given by (1.4), be the corresponding power series. Let F be the continued fraction corresponding to $p_0(x)$. By the first half of the theorem, $p_0(x)$ corresponds to F. It follows at once that (1.1) and F are identical since $p_0(x)$ determines uniquely its corresponding continued fraction. The proof of the theorem is complete.

We have seen that if the continued fraction terminates, the corresponding power series represents a rational function, by definition. Conversely, let $p_0(x)$ be a power series (1.4) which represents a rational function near the origin. We shall show that the corresponding continued fraction terminates. The case $p_0(x)\equiv 1$ is trivial. Suppose $p_0(x)\not\equiv 1$ and that the rational function it represents has the form

$$p_0(x) \equiv \frac{1 + e_1x + e_2x^2 + \cdots + e_rx^r}{1 + g_1x + g_2x^2 + \cdots + g_kx^k},$$

where at least one of the nonnegative integers r and k is greater than or equal to 1. One sees that $p_1(x)$ is of the form

$$p_1(x) \equiv \frac{1 + g_1x + g_2x^2 + \cdots + g_kx^k}{1 + h_1x + h_2x^2 + \cdots + h_mx^m},$$

where $0\leqq m<r$. If now $p_1(x)\equiv 1$, the continued fraction terminates Otherwise, form

$$p_2(x) \equiv \frac{1 + e_1'x + e_2'x^2 + \cdots + e_{r'}'x^{r'}}{1 + g_1'x + g_2'x^2 + \cdots + g_{k'}'x^{k'}}.$$

It is easy to see that $r'=m<r$ and $k'<k$. Since all $p_i(0)=1$, it follows readily from a repetition of this process that for some integer n, $p_n(x)\equiv 1$. The following theorem is therefore proved.

THEOREM 1.2. *A necessary and sufficient condition that a power series* (1.4) *represent a rational function near the origin is that its corresponding continued fraction terminate.*

The following theorems can now be proved in a manner precisely analogous to that of Perron [3, p. 303 ff.] for seminormal continued fractions; hence the proofs are omitted.

THEOREM 1.3. *Two finite or infinite continued fractions*

$$1 + \frac{a_1x^{\alpha_1}}{1} + \frac{a_2x^{\alpha_2}}{1} + \cdots, \qquad 1 + \frac{b_1x^{\beta_1}}{1} + \frac{b_2x^{\beta_2}}{1} + \cdots$$

have the same corresponding power series if and only if $a_i=b_i$ *and* $\alpha_i=\beta_i$, $(i=1, 2, \cdots)$.

THEOREM 1.4. *The terminating continued fractions*

$$1+\frac{a_1x^{\alpha_1}}{1}+\cdots+\frac{a_nx^{\alpha_n}}{1},\qquad 1+\frac{b_1x^{\beta_1}}{1}+\cdots+\frac{b_mx^{\beta_m}}{1}$$

represent the same rational function of x, *if and only if* $m=n$, $a_i=b_i$, $\alpha_i=\beta_i$, $(i=1, 2, \cdots, n)$.

This last theorem is a direct consequence of Theorem 1.3. The following theorem can be proved by means of the Weierstrass double series theorem. The proof will be omitted since it follows very closely the proof given by Perron [3, p. 342] in the semi-normal case.

THEOREM 1.5. *If the nonterminating continued fraction* (1.1) *converges uniformly in a closed region* T *containing the origin in its interior, it represents a regular analytic nonrational function of* x *throughout the interior of* T. *Further, the corresponding power series converges to the same function in and on the boundary of the largest circle which can be drawn with its center at the origin, lying wholly within* T.

It is clear that any function analytic in a neighborhood of the origin will have a formal expansion into a continued fraction (1.1). Examples show that the expansion does not always converge to the function in some neighborhood of the origin (Perron [3, p. 354]). On the other hand, convergent continued fractions of the form (1.1) may exist corresponding to power series with radius of convergence zero, and these afford a valuable means of summing such series (cf. Knopp [2, p. 554]; Borel [1, p. 55 ff.]).

If the power series $p_0(x)$ converges uniformly in some region containing the origin, we shall designate by $f_0(x)$ the analytic function to which it converges. We proceed with the following result.

THEOREM 1.6. *A necessary and sufficient condition that the continued fraction* (1.1) *converge uniformly to* $f_0(x)$ *in an open region* T *containing the origin is that the approximants* $A_n(x)/B_n(x)$ *be uniformly bounded in* T *for* n *sufficiently large.*

The proof of the necessity of the condition of the theorem is immediate.

Let* $|x|<R$ be a circular region in T, and set

$$f_n(x) = A_n(x)/B_n(x) = c_{n0}+c_{n1}x+\cdots+c_{nk}x^k+\cdots.$$

* The following method of proof was kindly suggested by the referee. It is distinctly less labored than the original proof constructed by the authors.

Since $|f_n(x)| \leqq M$ in $|x| < R$, we have by Cauchy's inequality $|c_{nk}| \leqq MR^{-k}$. Further, by a well known formula (Perron [3, p. 17]),

$$f_m - f_n = \frac{(-1)^n a_1 a_2 \cdots a_{n+1} x^{s_{n+1}} B_{m-n-1,n+1}}{B_m B_n}, \qquad m > n, \tag{1.8}$$

where $B_{\nu,\lambda}$ is the denominator of the νth approximant of the continued fraction $1 + K_{n=1}^{\infty}(a_{n+\lambda} x^{\alpha_{n+\lambda}}/1)$. Since $B_m(0) = B_n(0) = B_{\nu,\lambda}(0) = 1$, it follows that

$$f_m - f_n = g_t x^t + g_{t+1} x^{t+1} + \cdots, \qquad t = s_{n+1},$$

where $|g_t| \leqq 2MR^{-t}$. Hence

$$|f_m - f_n| \leqq 2M[(r/R)^t + (r/R)^{t+1} + \cdots], \quad |x| \leqq r < R, m > n.$$

It follows that $\lim_{n\to\infty} f_n(x)$ exists uniformly in $|x| \leqq r$, and by an application of Theorem 1.5 and the theorem of Stieltjes-Vitali (Titchmarsh [5, p. 168]) that

$$\lim_{n\to\infty} f_n(x) = f_0(x)$$

throughout T. The proof of the theorem is complete.

We note the following result.

THEOREM 1.7. *Two infinite subsequences of approximants of* (1.1) *which converge uniformly in a region T containing the origin converge to the same analytic function in T.*

Let $\{f_{n_k}(x)\}$ and $\{f_{m_k}(x)\}$ be two subsequences of approximants of (1.1) which converge uniformly in T. Then

$$\lim_{k\to\infty} f_{n_k}(x) = \phi(x), \qquad \lim_{k\to\infty} f_{m_k}(x) = \phi_1(x)$$

in T, where $\phi(x)$ and $\phi_1(x)$ are analytic in T. By (1.8)

$$\begin{aligned} |\phi(x) - \phi_1(x)| &= \lim_{k\to\infty} |f_{n_k}(x) - f_{m_k}(x)| \\ &= \lim_{k\to\infty} \left| \frac{a_1 a_2 \cdots a_{\nu_k+1} x^{s_{\nu_k+1}} B_{\mu_k-\nu_k-1-1,\nu_k}(x)}{B_{m_k}(x) B_{n_k}(x)} \right|, \end{aligned}$$

where μ_k is the larger and ν_k the smaller of n_k and m_k. Let $|x| < R$ be any circle in T. In the circle K: $|x| \leqq r < R$, making use of an argument used in the proof of the preceding theorem, we have

$$|\phi(x) - \phi_1(x)| = \lim_{k\to\infty} 2M[(r/R)^t + (r/R)^{t+1} + \cdots] = 0, \quad t = s_{\nu_k+1}$$

in K and hence in T. The proof is complete.

2. **Seminormal power series.** A power series

$$1+\sum_{i=1}^{\infty} c_i x^i \tag{2.1}$$

is called *seminormal* if all of the determinants

$$\phi_n = \begin{vmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_2 & c_3 & c_4 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n-1} \end{vmatrix}, \quad n = 1, 2, 3, \cdots,$$
$$\psi_n = \begin{vmatrix} c_2 & c_3 & c_4 & \cdots & c_n \\ c_3 & c_4 & c_5 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n-1} \end{vmatrix}, \quad n = 2, 3, 4, \cdots, \tag{2.2}$$

are different from zero. The corresponding continued fraction then takes the form (Perron [**3**, p. 304])

$$1+\frac{a_1x}{1}+\frac{a_2x}{1}+\cdots+\frac{a_nx}{1}+\cdots, \tag{2.3}$$

where

$$\phi_0 = 1, \qquad \psi_1 = 1, \qquad a_1 = \phi_1,$$
$$a_{2n+1} = -\psi_n\phi_{n+1}/\psi_{n+1}\phi_n,\ a_{2n} = -\phi_{n-1}\psi_{n+1}/\psi_n\phi_n,\ n = 1, 2, \cdots. \tag{2.4}$$

It is clear from (2.2) that (2.1) can fail to be seminormal if it has a single sufficiently large gap in its coefficients. For example, the gap in the coefficients of a power series whose first two terms are $1+kx^n$, $(k\neq 0)$, is too large to permit seminormality if $n=2, 3, \cdots$. The power series

$$c_0+\sum_{i=1}^{\infty} c_{k_i}x^{k_i}, \qquad c_{k_i}\neq 0.$$

where $k_1<k_2<k_3<\cdots$ and the k_i are positive integers, is said to have *Hadamard gaps*, if for infinitely many indices i

$$k_{i+1} > (1+\theta)k_i, \tag{2.5}$$

where θ is positive and fixed.

We come to the rather surprising result:

A seminormal power series may have Hadamard gaps.

This result will be established by exhibiting an example.

Consider the power series

$$1 + \sum_{i=0}^{\infty} c_{2^i} x^{2^i}, \tag{2.6}$$

where $c_{2^i} \neq 0$, $(i=0, 1, 2, \cdots)$. In order to show that the power series (2.6) is semi-normal, it must be shown that the determinants (2.2) are all different from zero. The elements c_k where $k \neq 2^i$, $(i=0, 1, 2, \cdots)$, are all zero.

If $n=2^i$, both determinants are essentially diagonal determinants and

$$\phi_{2^i} = \pm [c_{2^i}]^{2^i}, \qquad \psi_{2^i} = \pm [c_{2^i}]^{2^i-1}, \qquad i = 1, 2, \cdots. \tag{2.7}$$

Neither of these determinants is zero.

If $n=2^i+j$ where j is an integer such that $0<j<2^i$, the largest index of any c_λ appearing in ϕ_n is $2^{i+1}+2j-1$, where we observe that $2^{i+1}+2j-1<2^{i+1}+2^{i+1}-1=2^{i+2}-1$. Thus $c_{2^{i+1}}$ appears in the last $2j$ rows and the last $2j$ columns of ϕ_n. Also, no nonzero elements other than $c_{2^{i+1}}$ appear in the last j rows or the last j columns. ϕ_n is now expanded by Laplace's method in terms of the $2j$-rowed minors in the last $2j$ columns, together with their complementary minors. The value of the minor formed from the last $2j$ rows is $(-1)^j[c_{2^{i+1}}]^{\nu j}$, and its complementary minor is ϕ_{2^i-j}. Any other $2j$-rowed minor which is not zero must have at least one nonzero element in each row and in each column; hence at least one of the last j rows of ϕ_n is not used in such a minor. Then the complementary minor must have at least one row whose elements are all zero. Finally,

$$\phi_n = (-1)^j[c_{2^{i+1}}]^{2j}\phi_{2^i-j}, \quad i = 1, 2, \cdots; j = 1, 2, \cdots, 2^i-1. \tag{2.8}$$

A similar expansion of ψ_n by $(2j-1)$-rowed minors in the last $(2j-1)$ columns gives

$$\begin{aligned} \psi_n &= (-1)^{j-1}[c_{2^{i+1}}]^{2j-1}\psi_{2^i-j+1}, \\ & \qquad i = 1, 2, \cdots; j = 1, 2, \cdots, 2^i-1. \end{aligned} \tag{2.9}$$

Since all of the coefficients c_{2^i} are different from zero, it follows that all of the ϕ- and ψ-determinants are different from zero, and the power series (2.6) is seminormal.

It is easily seen that the power series has Hadamard gaps since $2^{i+1}>(1+\theta)2^i$ for any positive value of $\theta<1$.

The power series (2.6) is in a sense a limiting form for seminormal power series. In fact, no seminormal power series has more zero coefficients, with indices not exceeding a given positive integer, than has the power series (2.6). This is a consequence of the following theorem.

THEOREM 2.1. *A necessary condition that the power series*

$$c_0 + \sum_{i=0}^{\infty} c_{n_i} x^{n_i},$$

where $c_{n_i} \neq 0$, $(i = 0, 1, 2, \cdots)$, *be seminormal is that*

$$n_0 = 1, \qquad n_i < n_{i+1} \leqq 2n_i.$$

It has already been noted that the power series is not seminormal if $n_0 > 1$. If $n_{i+1} > 2n_i$, it is seen from (2.2) that at least one of the ϕ- and ψ-determinants is zero, and the power series cannot be seminormal in this case. The proof is complete.

We now develop the continued fraction (2.3) corresponding to the power series (2.6). On applying (2.4), (2.7), (2.8), and (2.9) we find that the numbers a_i are given by the following formulas:

$$(2.10)\quad \begin{aligned} &a_{2^{(i+1)}+2j} = a_{2^{(i+1)}-2j+1}, \qquad a_{2^{(i+1)}+2j+1} = a_{2^{(i+1)}-2j}, \\ &\qquad\qquad j = 1, 2, \cdots, 2^i - 1;\ i = 1, 2, \cdots, \\ &a_{2^{(i+1)}+1} = -c_1 c_{2^{(i+1)}}/[c_{2^i}]^2, \qquad a_{2^{(i+1)}} = c_1 c_{2^{(i+1)}}/[c_{2^i}]^2, \\ &\qquad\qquad i = 1, 2, \cdots. \end{aligned}$$

These recurrence formulas together with the initial values

$$(2.11)\qquad a_1 = c_1, \qquad a_2 = -c_2/c_1, \qquad a_3 = c_2/c_1$$

completely determine the coefficients of the continued fraction.

It is not difficult to show that

$$(2.12)\qquad a_{2n} = -a_{2n+1}, \qquad n = 1, 2, \cdots.$$

But condition (2.12) is precisely a necessary and sufficient condition that the corresponding seminormal power series of the continued fraction have no odd powers of x except the first power (Perron [**3**, p. 336]).

If the coefficients of the power series (2.6) are all equal to $c \neq 0$, the coefficients of the corresponding continued fraction are all 1 and -1, except the first which is c.

3. An important special case. In this section we discuss, as an example, continued fractions (1.1) subject to the condition that the series $\sum_{i=1}^{\infty} |a_i x^{\alpha_i}|$ converge in an open circle T containing the origin.

THEOREM 3.1. *If the series* $\sum_{i=1}^{\infty} a_i x^{\alpha_i}$ *converges absolutely throughout an open region* T *containing the origin, the continued fraction* (1.1) *converges to a function regular in* T *except possibly for poles. This function is regular near the origin.*

Using a method strictly analogous to one of von Koch (Perron [**3**, p. 345]) one can show that in any circle $|x| \leqq r$ lying wholly within T

$$\lim_{n\to\infty} B_n(x) = B(x), \qquad \lim_{n\to\infty} A_n(x) = A(x)$$

uniformly in $|x| \leqq r$. Hence $A(x)$ and $B(x)$ are regular in T. One then shows, following Maillet (Perron [**3**, p. 346]), that $A(x)$ and $B(x)$ have no common zeros in T. The proof may thus be completed.

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