fined by (3.11) admits solutions $H(x^i, y)$ of the equations (3.7) other than $H=x^n$. By methods similar to those hitherto employed, we find that the most general solution for H of the form $H=H(x^n, y)$ is

$$H = \alpha(y)\cdot x^n + \beta(y), \qquad c = 0, \tag{3.13}$$

or

$$H = \alpha(y)\cdot x^n, \qquad c \neq 0, \tag{3.14}$$

where $\alpha(y)$ and $\beta(y)$ are arbitrary functions of y. We note that the E_{n+1} obtained by using the H defined by (3.14) coincides with (3.12).

It can be shown that solutions for H which involve some of the x^ν do not exist unless the E_n defined by (3.11) may be mapped conformally on another Einstein space in more than one way. Hence, if this is not the case, the E_n's may only be imbedded in the unique E_{n+1} defined by (3.12) if $c\neq 0$ and only in the E_{n+1}'s defined by (3.1), (3.11), and (3.13) if $c=0$. In this last case, $a=b=0$.

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CONCERNING THE BOUNDARY OF A COMPLEMENTARY DOMAIN OF A CONTINUOUS CURVE*

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Much study by various investigators has been given to the nature of the boundary of a complementary domain of a locally compact continuous curve in the plane and in certain other spaces.† It is the purpose of this paper to continue this investigation in less restricted spaces which satisfy the Jordan curve theorem and to establish certain results (from which many of the known results follow immediately) in such a way as to bring out what is essential for their validity.

It is first necessary to establish the following lemma.

Lemma A. *If a locally compact nondegenerate continuous curve M in a complete Moore space contains no simple triod, then M is a simple continuous curve.*‡

* Presented to the Society, December 30, 1938.

† See the bibliography and Chapter 4 of R. L. Moore's *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. Hereinafter, this book will be referred to as *Foundations*, and the reader is referred to it for many theorems and the definition of certain terms and phrases used in this paper.

‡ A *complete Moore space* is a space satisfying Axioms 0 and 1 of *Foundations*. A *simple continuous curve* is either a simple continuous arc, a simple closed curve, an open curve, or a ray.

PROOF. With the help of Theorems 118 and 120 in Chapter 1 and the arguments for Theorems 6 and 7 in Chapter 2 of *Foundations*, it is easy to see that if "point" is interpreted to mean a point of M and "region" is interpreted to mean an open subset of M, then as a space, M satisfies Axioms 0–2 of *Foundations* and "limit point" is invariant. Suppose that H is any compact subcontinuum of M. By Theorem 65 in Chapter 1 of *Foundations*, H contains two distinct points, A_1 and A_2, which are non-cut points of H. Suppose that P is a point of H distinct from A_1 and A_2 which is a boundary point of H with respect to M. Let D denote a connected open subset of M containing P such that $\overline{D}$ contains neither A_1 nor A_2. The set D contains a point A_3 of $M-H$. Let d_1 and d_2 denote mutually exclusive connected open subsets of M containing A_1 and A_2, respectively, but containing no point of D. For each point X of $H-(A_1+A_2)$, there exists a connected open subset d_X of M containing X but not containing A_1, A_2, or A_3. Let d denote $\sum d_X$ for all points X of $H-(A_1+A_2)$. Then since $H-A_2$ is connected, $D+d_1+d$ is a connected open subset of $M-A_2$ containing A_1 and A_3. By Theorem 1 in Chapter 2 of *Foundations*, there exists an arc T_1 in $D+d_1+d$ from A_1 to A_3. Since $H-A_1$ is connected, d_2+d is a connected open subset of M containing $H-A_1$ and a point of T_1. Let T_2 denote an arc in d_2+d irreducible from A_2 to T_1. Then T_1+T_2 is a simple triod contrary to hypothesis. Hence no point of H distinct from A_1 and A_2 is a boundary of H with respect to M, and consequently no compact subcontinuum of M has more than two boundary points with respect to M. Then by Theorem 20′ in Chapter 2 of *Foundations*, M is a simple continuous curve.

COROLLARY. *If a compact nondegenerate continuous curve in a complete Moore space contains no simple triod, then it is either an arc or a simple closed curve.*†

The results of this paper that follow assume that Axioms 0–4 of *Foundations* hold true. Let S denote the set of all points.

THEOREM 1. *Suppose that K is a locally compact continuum lying in the boundary of a connected domain D. Then in order that K be a continuous curve it is necessary and sufficient that K be a subset of a continuous curve M which contains no point of D.*

PROOF. The necessity is obvious. It remains only to prove the sufficiency. Suppose, on the contrary, that K is not connected im kleinen at a point O of K. Then there exists a domain Q_1 containing O such

† Cf. Theorem 71 in Chapter 4 of *Foundations*.

that $Q_1 \cdot K$ is compact and the component of $Q_1 \cdot K$ which contains O is not open with respect to K at O. Let Q_2 denote a domain containing O and lying together with its boundary in Q_1. There exist two sequences of points $O_1, O_2, O_3, \cdots$ and $P_1, P_2, P_3, \cdots$ such that (1) for each n, $(n=1, 2, 3, \cdots)$, O_n and P_n belong to the same component of $Q_2 \cdot K$, (2) if $m \neq n$, then O_m and O_n belong to different components of $Q_1 \cdot K$, and (3) $O_1, O_2, O_3, \cdots$ converges to O and $P_1, P_2, P_3, \cdots$ converges to a point P of $Q_1 \cdot K$ distinct from O. Let U and V denote two connected open subsets of M containing O and P, respectively, such that $\overline{U} \cdot \overline{V} = 0$. By (3) there exist three integers n_1, n_2, and n_3 such that O_{n_1}, O_{n_2}, O_{n_3} and P_{n_1}, P_{n_2}, P_{n_3} lie in U and V, respectively. For each i, $(i=1, 2, 3)$, let T_i denote the component of $Q_2 \cdot K$ that contains $O_{n_i} + P_{n_i}$. By (2), $\overline{T}_i \cdot \overline{T}_j = 0$ if $i \neq j$. Hence, $\overline{T}_1$, $\overline{T}_2$, and $\overline{T}_3$ are mutually exclusive compact continua lying in $Q_1 \cdot K$, and each contains both a point of U and a point of V. For each i, $(i=1, 2, 3)$, there exists a finite collection H_i of sets such that (a) each element of H_i is a connected open subset of M containing a point of $\overline{T}_i$, (b) H_i covers $\overline{T}_i$, (c) no element of H_i contains a point of an element of H_j if $i \neq j$, and (d) no element of H_i contains both a point of U and a point of an element of H_i containing a point of V. The set H_i^* is a connected domain with respect to M containing $\overline{T}_i$.† By Theorem 10 in Chapter 2 of *Foundations*, there exists, for each i, $(i=1, 2, 3)$, an arc Y_iZ_i in H_i^* from a point Y_i of U to a point Z_i of V. Let Y_1Y_2 and Y_1Y_3 denote arcs lying in U and Z_1Z_2 and Z_1Z_3 denote arcs lying in V, with end points as indicated. Then there exist three arcs AW_1B, AW_2B, and AW_3B from a point A of U to a point B of V lying in $Y_1Y_2 + Y_1Z_1 + Z_1Z_2$, $Y_1Y_2 + Y_2Z_2 + Z_1Z_2$, and $Y_1Y_3 + Y_3Z_3 + Z_1Z_3$, respectively, which have only their end points in common. Let ω, "the point at infinity," be a point of D. From Theorems 4 and 5 in Chapter 3 of *Foundations*, it follows that the sum of one pair of these arcs, say $AW_1B + AW_3B$, forms a simple closed curve J whose interior contains the other arc less its end points, that is, $AW_2B - (A+B)$. By (d) above, it is clear that some element d of H_2 contains a point of AW_2B and a point X of T_2 but no point of $U+V$. By (c), d contains no point of $H_1^* + H_3^*$. Consequently, d contains no point of J since J lies in $U + H_1^* + V + H_3^*$. But X is a boundary point of D. Hence $D+d$ is a connected set containing no point of J but containing ω and a point of AW_2B in the interior of J, which is a contradiction.

EXAMPLE. *Theorem* 1 *is false if the stipulation that* K *be locally com-*

† H_i^* denotes the sum of the elements of H_i.

pact is omitted. This may be seen in an example discovered by R. L. Moore some years ago but as yet unpublished. This example may be roughly described as follows. In a euclidean 3-space for each n, $(n=1, 2, 3, \cdots)$, let U_n denote a circular cylinder whose radius is

$$\frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

and whose axis is a line parallel to the z axis passing through $(1/n, 0, 0)$. Let S denote the set of all points P such that either (1) P is in the xy-plane but is not, for any n, $(n=1, 2, 3, \cdots)$, within U_n, or (2) P is in the plane $z=1$ and is, for some integer n, within U_n, or (3) P is, for some integer n, in U_n and either in or between the planes $z=0$ and $z=1$. If "limit point" is given the ordinary 3-dimensional sense, S satisfies Axioms 0-5 of *Foundations*. Let K denote the intersection of the xz-plane with S; let M denote all of the points of S either on or between the two planes $y=0$ and $y=1$; and let D denote the component of $S-M$ which contains $(0, -1, 0)$. Then M is a continuous curve in S and K is a continuum in S lying both in M and in the boundary of M. But obviously K is not connected im kleinen at $(0, 0, 0)$.

This example should be remembered in connection with certain results to follow—Theorem 8, in particular.

Theorem 1 establishes the truth of the following two theorems.

THEOREM 2. *Every component of the boundary of a complementary domain of a locally compact continuous curve is a locally compact continuous curve.*

THEOREM 3. *If the boundary of a complementary domain of a locally compact continuous curve is connected, it is itself a locally compact continuous curve.*†

THEOREM 4. *If D and Q are two mutually exclusive connected domains whose boundaries contain a simple closed curve J, then J separates D from Q.*

PROOF. Suppose, on the contrary, that D and Q both lie in I, one of the complementary domains of J. Let ω, "the point at infinity," be a point in the other complementary domain of J. Then I is the interior of J. There exists in D an arc segment T whose end points, A and B, lie on J.‡ There exist two points, C and F, of J which are

† Cf. Theorem 40 in Chapter 4 of *Foundations.*

‡ A connected open subset of non-end points of a simple continuous curve is called a *segment.* An *arc segment* is a segment of an arc.

separated on J by A and B. Let I_1 and I_2 denote the interiors of ACB (of J)$+T$ and AFB (of J)$+T$, respectively. By Theorem 4 in Chapter 3 of *Foundations*, $I=T+I_1+I_2$. Since Q lies in I and contains no point of T, Q is a subset either of I_1 or of I_2. If Q lies in I_1, then F is not in the boundary of Q, and if Q lies in I_2, then C is not in the boundary of Q. In either case, some point of J is not in the boundary of Q which is a contradiction.

THEOREM 5. *If D is a connected domain and E is a point of $S-\overline{D}$, then the outer boundary of D with respect to E is either acyclic or a simple closed curve.*†

PROOF. Suppose that the boundary of Q, the component of $S-\overline{D}$ which contains E, contains a simple closed curve J. Then J is in the boundary of both D and Q. If the boundary of Q contains a point P not in J, then P is obviously in the boundary of D and $D+P+Q$ is a connected subset of $S-J$. But this contradicts Theorem 4. Hence J is the complete boundary of Q.

THEOREM 6. *If D is a complementary domain of a locally compact continuous curve M, and E is a point of $S-\overline{D}$, then every component of the outer boundary of D with respect to E is a continuous curve.*

PROOF. Let C denote a component of the boundary of Q, the complementary domain of $\overline{D}$ which contains E. Then C is a subset of a component K of the boundary of D. By Theorem 2, K is a locally compact continuous curve. But K contains no point of Q; hence, by Theorem 1, C is a continuous curve.

THEOREM 7. *If D is a complementary domain of a locally compact continuous curve M, and E is a point of $S-\overline{D}$, then every component of the outer boundary of D with respect to E is atriodic.*

PROOF. Let C denote a component of the boundary of Q, the complementary domain of $\overline{D}$ which contains E. Suppose that C contains a triod. Using the preceding theorem and Theorem 10 in Chapter 2 of *Foundations*, it can be shown that C contains three arcs A_1O, A_2O, and A_3O which have only the point O in common. Let d_1, d_2, and d_3 denote three mutually exclusive connected open subsets of M containing A_1, A_2, and A_3, respectively, but not containing O. With the help of Theorems 1, 2, and 10 of Chapter 2 of *Foundations*, it is easy to see that, for each i, $(1\leqq i\leqq 3)$, there exists in $Q+d_i$ an arc PA_i' from a point P of Q to a point A_i' of A_iO such that P is the only point

† Cf. Theorem 41 in Chapter 4 of *Foundations*.

that PA_1', PA_2', and PA_3' have in common. Let ω, "the point at infinity," be a point of D, and for each i, $(1 \leqq i \leqq 3)$, let $PA_i'O$ denote PA_i' plus the interval of A_iO from A_i' to O. It follows from Theorems 4 and 5 in Chapter 3 of *Foundations* that the sum of one pair of these arcs, say $PA_1'O+PA_3'O$, is a simple closed curve J whose interior I contains the internal points of the other arc, $PA_2'O$. Since J contains no point of D, I contains no point of D. But I contains a boundary point A_2' of D, which is a contradiction.

THEOREM 8. *If D is a complementary domain of a locally compact continuous curve M, and E is a point of $S-\overline{D}$, then every nondegenerate component of the outer boundary of D with respect to E is a simple continuous curve.*†

Theorem 8 follows from Lemma A and Theorems 6 and 7.

THEOREM 9. *If* (1) *D is a complementary domain of a locally compact continuous curve M,* (2) *E is a point of $S-\overline{D}$,* (3) *C is a component of β, the outer boundary of D with respect to E, and* (4) *X is a non-end point of C, then X is not a limit point of $\beta - C$.*

PROOF. Let Q denote the component of $S-\overline{D}$ which contains E; and suppose that the theorem is false. Then there exists an arc A_1XA_2 which contains X as an internal point and a connected open subset d_X of M containing X but containing neither A nor B. The set d_X contains a point A_3 of $\beta - C$. Let A_3O denote an arc in d_X irreducible from A_3 to a point O of A_1XA_2, and let A_1O and A_2O denote the intervals of A_1XA_2 from A_1 and A_2, respectively, to O. Let d_1, d_2, and d_3 denote three mutually exclusive connected open subsets of M containing A_1, A_2, and A_3, respectively, but not containing O. With the help of Theorems 1, 2, and 10 in Chapter 2 of *Foundations*, it is easy to see that for each i, $(1 \leqq i \leqq 3)$, there exists an arc PA_i' lying in $Q+d_i$ which is irreducible from a point P of Q to A_iO such that P is the only point that PA_1', PA_2', and PA_3' have in common. For each i, $(1 \leqq i \leqq 3)$, let $PA_i'O$ denote PA_i' plus the interval of A_iO from A_i' to O, and let A_iA_i' denote the interval of A_iO from A_i to A_i'. Let ω, "the point at infinity," be a point of D. It follows from Theorems 4 and 5 in Chapter 3 of *Foundations* that the sum of one pair, say $PA_1'O+PA_3'O$, of the arcs $PA_1'O$, $PA_2'O$, and $PA_3'O$ is a simple closed curve J whose interior I contains the internal points of the other arc, $PA_2'O$. Since J contains no point of D, I contains no point of D. But A_2A_2' contains an internal point A_2' of $PA_2'O$ and

† Cf. Theorem 41 in Chapter 4 of *Foundations.*

contains no point of J. Hence I contains A_2, a boundary point of D, which is a contradiction.

THEOREM 10. *If D is a complementary domain of a compact continuous curve and E is a point of $S-\overline{D}$, then the outer boundary of D with respect to E is either a simple closed curve or the sum of the elements of a countable collection G of mutually exclusive arcs and a totally disconnected closed point set H such that $G^*\cdot H$ is the set of all end points of the arcs of G.*

Theorem 10 follows from Theorems 5, 8, and 9.

THEOREM 11. *If (1) D is a complementary domain of a locally compact continuous curve M, (2) E is a point of $S-\overline{D}$, (3) C is a component of β, the outer boundary of D with respect to E, and (4) β and the boundary of D are identical, then D plus the non-end points of C is a connected, connected im kleinen, inner limiting set.*†

PROOF. Let H denote D plus the non-end points of C. Obviously, H is a connected inner limiting set and is connected im kleinen at all of the points of D. Suppose that X is a non-end point of C. Then from Theorems 8 and 9 it is easy to see that there exists a region R which contains X but contains neither an end point of C nor any point of $\beta-C$. Let T denote the component of $R\cdot C$ which contains X, and let R_1 denote the component of $R-R\cdot(C-T)$ which contains C. Obviously T is a segment, R_1 is a domain, and since any point of $R_1\cdot H$ may be joined to T by an arc in R_1, $R_1\cdot H$ is a connected subset of R which is open with respect to H. Since this is true for any region R containing X which contains neither an end point of C nor any point of $\beta-C$, H is connected im kleinen at X.

THEOREM 12. *If K is the boundary of a complementary domain D of a locally compact continuous curve M, β is the outer boundary of D with respect to a point E of $S-\overline{D}$, and H is a component of $K-\beta$, then $\overline{H}$ is a continuous curve having only one point in β.*‡

PROOF. Let Q denote the component of $S-\overline{D}$ which contains E. Then β is the boundary of Q. Evidently $\overline{H}$ contains at least one point of β. Suppose that $\overline{H}$ contains two points, A_1 and A_2, of β. Let d_1 and d_2 denote two mutually exclusive connected open subsets of M containing A_1 and A_2, respectively. By Theorems 2 and 10 in Chapter 2 of *Foundations*, $H+d_1+d_2$ contains an arc T_H from A_1 to A_2.

† Cf. Theorem 18 in Chapter 3 of *Foundations.*

‡ Cf. Theorem 43 in Chapter 4 of *Foundations.*

By Theorems 1 and 2 in Chapter 2 of *Foundations*, it follows that $Q+d_1+d_2$ contains an arc T_Q irreducible from $T_H \cdot d_1$ to $T_H \cdot d_2$. Then T_H+T_Q contains a simple closed curve J which contains a point X of H and a point Y of Q but which contains no point of D. Let I denote the complementary domain of J which contains no point of D. The domain I contains no point of $\overline{D}$ and consequently no point of β. Hence $I+X+Y$ is a connected point set lying in Q since it contains a point of Q but no point of β. But since X is a point of $\overline{D}$, this is a contradiction. Consequently $\overline{H}$ has only one point O in β. Furthermore, it is easy to see that $\overline{H}$ is connected im kleinen; for it is evident that $\overline{H}$ is connected im kleinen at all of its points except possibly O, and if d is any connected open subset of M containing O, the components of $d-O$ which contain points of H together with O form a connected open subset of $\overline{H}$.

However, despite Theorems 8, 9, and 12, Theorem 11 is false if condition (4) is omitted. Speaking roughly, condition (4) may be omitted and the theorem remain true if S does not contain both "hills" and "holes."

The reader should note that Theorems 6 to 12 inclusive remain true if, instead of postulating that D is a complementary domain of a locally compact continuous curve, it is postulated that D is a complementary domain of a continuous curve and the boundary of D is locally compact. This is quite evident since the property of local compactness is not used in any proof other than the proof of Theorem 1. Of course, the boundaries of the domains involved must be locally compact in order to make the use of Theorem 1 valid.

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