TOTALLY GEODESIC EINSTEIN SPACES*

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1. **Introduction.** An Einstein space† E_m is defined as a Riemann space V_m whose mean curvature a is a constant at each point; that is,‡

$$(1.1) \qquad R_{\alpha\beta} = -ag_{\alpha\beta}$$

where $R_{\alpha\beta}$ and $g_{\alpha\beta}$ are the Ricci and metric tensors of V_m, respectively. We suppose that the dimension m of E_m exceeds 3. For every surface is an E_2 and the only E_3's are the spaces of constant curvature. In both these cases, the discussion which parallels that given in this note is obvious and trivial. Since $m>3$, it is a well known consequence of (1.1) that a is a constant throughout the space. In this note, we discuss the properties of an E_m which admits families of totally geodesic subspaces which are also Einstein spaces. It is shown that this subject is closely related to the problems of finding (a) all Einstein spaces which may be imbedded as hypersurfaces of a space of constant curvature (b) Einstein spaces which are conformal to Einstein spaces. In a restricted sense,§ we also find the first fundamental form of E_m. It is assumed that the first fundamental forms of E_m and of its subspaces which are discussed below are nonsingular although they may be indefinite.

2. **Separable Einstein spaces.** It has been shown by Bompiani‖ that the necessary and sufficient condition that the subspaces x^p = const. and the orthogonal subspaces x^i = const. be totally geodesic in V_m is that

$$(2.1) \qquad g_{ij} = f_{ij}(x^k), \qquad g_{pq} = h_{pq}(x^r), \qquad g_{ip} = 0.$$

When the first fundamental form of V_m satisfies (2.1), it is called

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† We represent an m-dimensional Riemann space and Einstein space by V_m and E_m, respectively.

‡ Throughout this note, $\alpha, \beta, \gamma, \delta$; h, i, j, k; p, q, r; λ, μ, ν have the ranges $1, 2, \cdots, m$; $1, 2, \cdots, n$; $n+1, n+2, \cdots, m$; $1, 2, \cdots, n-1$, respectively. An index which appears twice in an expression is to be summed over the appropriate range. A free index of a tensor equation assumes each value of its range.

§ The first fundamental form of E_m is obtained in a preferred coordinate system and depends upon the unknown first fundamental form of an arbitrary Einstein space.

‖ E. Bompiani, *Spazi Riemanniani luoghi di varietà totalmente geodetiche*, Rendiconti del Circolo Matematico di Palermo, vol. 48 (1924), p. 124.

separable, and the two forms $f_{ij}dx^i dx^j$ and $h_{pq}dx^p dx^q$ are called its *components*. It is clear that the components are the first fundamental forms of the totally geodesic V_n's and V_{m-n}'s of V_m and that the V_n's (as well as the V_{m-n}'s) are isometric.

We denote the Christoffel symbols of the first and second kind by $[\alpha\beta, \gamma]$, $\{\alpha|\beta\gamma\}$ for V_m; $\overline{[ij, k]}$, $\overline{\{i|jk\}}$ for V_n, and $\overline{[pq, r]}$, $\overline{\{p|qr\}}$ for V_{m-n}. Then it follows from (2.1) that

$$(2.2)\qquad [ij, k] = \overline{[ij, k]}, \qquad \{i|jk\} = \overline{\{i|jk\}},$$

$$(2.3)\qquad [pq, r] = \overline{[pq, r]}, \qquad \{p|qr\} = \overline{\{p|qr\}},$$

$$(2.4)\qquad [\alpha\beta, \gamma] = 0, \qquad \{\alpha|\beta\gamma\} = 0$$

if α, β, γ are not all in the same range. The Ricci tensor of V_m is defined as

$$(2.5)\qquad \begin{aligned} R_{\alpha\beta} = \frac{\partial^2 \log g^{1/2}}{\partial x^\alpha \partial x^\beta} - \frac{\partial}{\partial x^\gamma}\{\gamma|\alpha\beta\} + \{\gamma|\alpha\delta\}\{\delta|\gamma\beta\} \\ - \{\gamma|\alpha\beta\}\frac{\partial \log g^{1/2}}{\partial x^\gamma} \end{aligned}$$

where $g = |g_{\alpha\beta}|$.

We first suppose that $n>1$ and $m-n>1$. Then from (2.2), (2.3), (2.4), and (2.5),

$$(2.6)\qquad R_{ip} = 0, \qquad R_{ij} = \overline{R}_{ij}, \qquad R_{pq} = \overline{R}_{pq}$$

where $\overline{R}_{ij}$ and $\overline{R}_{pq}$ are the Ricci tensors of V_n and V_{m-n}, respectively. If V_m is an E_m, it follows from (1.1), (2.1), and (2.6) that

$$(2.7)\qquad \overline{R}_{ij} = -af_{ij}, \qquad \overline{R}_{pq} = -ah_{pq}.$$

Hence V_n and V_{m-n} are both Einstein spaces of the same mean curvature as E_m. Conversely, by reversing the proof we find that (1.1) is a consequence of (2.1) and (2.7). This proves the following theorem:

THEOREM 2.1. *Let the first fundamental form of an Einstein space of dimensionality $m>3$ of mean curvature a be separable into components whose dimensions exceed* 1. *Then each component is the first fundamental form of an Einstein space of mean curvature a. Conversely, only the first fundamental forms of Einstein spaces are separable in this manner.*

If $m=4$, $n=2$, it follows, from this theorem and the observation that a two-dimensional space of constant mean curvature has constant Riemann curvature, that each component is the first fundamental form of a space of constant curvature a. This was first proved by Kasner.* In recent papers,† we have shown that Einstein spaces which are proper hypersurfaces of any space of constant curvature are either spaces of constant curvature themselves or are separable. By a repeated application of Theorem 2.1, we may obtain an obvious generalization which applies to an E_m whose first fundamental form is separable into more than two components.

If $n>1$ and $m=n+1$, and if (2.1) is satisfied, the curves $x^i=\text{const.}$ are geodesics of E_m. In this case, equations (2.6) become

$$(2.8)\qquad R_{im} = 0, \qquad R_{ij} = \overline{R}_{ij}, \qquad R_{mm} = 0.$$

As a consequence of (1.1), (2.1), and (2.8), we find that $\overline{R}_{ij}=0$, $a=0$, and conversely. We have proved the following theorem:

THEOREM 2.2. *A V_{n+1} which admits ∞^1 parallel totally geodesic E_n's is an E_{n+1} if and only if the mean curvature of the E_n's is zero. In this case, the mean curvature of E_{n+1} is also zero.*

3. **E_{n+1} with totally geodesic E_n's.** We suppose $m=n+1$ and that E_{n+1} admits ∞^1 totally geodesic E_n's which are not parallel. Since the first fundamental form of the necessarily isometric E_n's is non-singular, it follows that the normals to the E_n's in E_{n+1} are not null vectors.‡ Hence, in accordance with a slightly weaker form of the theorem of Bompiani quoted above,

$$(3.1)\qquad g_{ij} = f_{ij}(x^k), \qquad g_{n+1,n+1} = eH^2(x^i, y), \qquad g_{i,n+1} = 0,$$

where $y=x^{n+1}$, e is $+1$ or -1, and $f_{ij}dx^idx^j$ is the first fundamental form of each E_n. Since the hypersurfaces are not parallel, it follows that $H(x^i, y)$ cannot be a function of y only but must involve the x^i. Because of (3.1), we find that (2.2) holds and (2.4) is true if one of α, β, γ is $n+1$ and the other two lie in the range 1, 2, $\cdots$, n. Also

$$(3.2)\qquad \begin{gathered} \{n+1 \mid i\ n+1\} = \frac{\partial \log H}{\partial x^i}, \quad \{n+1 \mid n+1\ n+1\} = \frac{\partial \log H}{\partial y}, \\ \{i \mid n+1\ n+1\} = -\,eH^2 \frac{\partial \log H}{\partial x^j}\cdot g^{ij}, \end{gathered}$$

* E. Kasner, *An algebraic solution of the Einstein equations*, Transactions of this Society, vol. 27 (1925), pp. 101–105, and *Separable quadratic differential forms and Einstein solutions*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 95–96.

† A. Fialkow, *Einstein spaces in a space of constant curvature*, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 30–34, and *Hypersurfaces of a space of constant curvature*, Annals of Mathematics, (2), vol. 39 (1938), pp. 762–785. The term "proper" is defined in these papers.

‡ L. P. Eisenhart, *Riemannian Geometry*, 1926, p. 144.

where g^{ij} are the contravariant components of g_{ij}. From (2.2), (2.4), (2.5), and (3.2),

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\begin{aligned}
R_{n+1,n+1} = & - \frac{\partial}{\partial x^i}\{i \mid n+1\ n+1\} \\
(3.3) \qquad & + \{n+1 \mid n+1\ i\}\{i \mid n+1\ n+1\} \\
& - \{i \mid n+1\ n+1\} \frac{\partial \log f^{1/2}}{\partial x^i},
\end{aligned}
$$

$$(3.4) \qquad R_{i,n+1} = 0,$$

$$(3.5) \qquad R_{ij} = \overline{R}_{ij} + (\log H)_{,ij} + (\log H)_{,i}(\log H)_{,j},$$

where the comma denotes covariant differentiation with respect to the form $f_{ij}dx^idx^j$, and $f = |f_{ij}|$. Since E_{n+1} and E_n are Einstein spaces, (1.1) and

$$(3.6) \qquad \overline{R}_{ij} = -bf_{ij}$$

are true.

It follows from (1.1), (3.1), and (3.6) that (3.5) becomes

$$(3.7) \qquad H_{,ij} = cHf_{ij}$$

where

$$(3.8) \qquad c = b - a.$$

The integrability conditions of (3.7) are

$$H_{,ijk} - H_{,ikj} = H_{,h}\overline{R}^h_{ijk}$$

which become, by virtue of (3.7),

$$H_{,h}\overline{R}^h_{ijk} = c(H_{,k}f_{ij} - H_{,j}f_{ik}).$$

The tensor $\overline{R}^h_{ijk}$ is the Riemann curvature tensor of E_n. If we multiply this equation by the contravariant components f^{ij} of the metric tensor of E_n and sum for i, j, we find after using (3.6) that

$$[c(n-1) + b]H_{,k} = 0.$$

Since $H_{,k}$ cannot be zero for every value of k, the last equation and (3.8) show that

$$(3.9) \qquad nb = (n-1)a$$

is a necessary condition that (3.7) have a solution.

Since $\partial \log (f)^{1/2}/\partial x^i = \{\overline{k \mid ki}\}$, after using (3.2), equation (3.3) becomes

$$R_{n+1,n+1} = eH\left[\frac{\partial}{\partial x^i}(f^{ij}H_{,j} + f^{ij}H_{,j}\{k\,|\,l\,i\}\right]$$

or

$$(3.10) \qquad R_{n+1,n+1} = eH\Delta_2 H$$

where $\Delta_2 H = f^{ij}H_{,ij}$. From (3.7), (3.8), (3.9), and (3.10),

$$R_{n+1,n+1} = -\,aeH^2.$$

According to (3.1), this equation and (3.4) both obey (1.1). Hence E_{n+1} will admit ∞^1 nonparallel totally geodesic E_n's if and only if a solution of (3.7) exists.

Since $H(x^i, y)$ is not independent of the x^i, we may choose coordinates x^i so that $H = x^n$ for some fixed value of y and such that $f_{n\lambda} = 0$. Then (3.7) becomes

$$-\,\{\overline{n\,|ij}\} = cx^n f_{ij}.$$

Now Brinkmann* has shown that E_n admits a solution of these equations if and only if its metric tensor satisfies

$$(3.11) \qquad \begin{aligned} f_{nn} &= (cx^{n^2} + d)^{-1}, \\ f_{\lambda\mu} &= (cx^{n^2} + d)F_{\lambda\mu}(x^\nu), \qquad f_{n\lambda} = 0, \qquad d \cdot \text{constant}, \end{aligned}$$

and the form $F_{\lambda\mu}(x^\nu)dx^\lambda dx^\mu$ is the first fundamental form of an E_{n-1}. According to Brinkmann, (3.11) is the necessary and sufficient condition that E_n be conformal to another Einstein space by means of a transformation $d\bar{s} = \sigma ds$ with $\Delta_1\sigma \neq 0$ where $\Delta_1\sigma = f^{ij}\sigma_{,i}\sigma_{,j}$. If we suppose $H = x^n$ for all values of y, using (3.1) we may write the first fundamental form of one E_{n+1} which satisfies the conditions of the problem as

$$(3.12) \qquad ds^2 = f_{ij}dx^i dx^j + ex^{n^2}dx^{n+1^2}$$

where the f_{ij} satisfy (3.11). This proves the following theorem:

THEOREM 3.1. *A one-parameter family of isometric E_n's may be imbedded as ∞^1 nonparallel totally geodesic hypersurfaces of an E_{n+1} if and only if each E_n may be mapped conformally on another Einstein space by means of a function σ with $\Delta_1\sigma \neq 0$. If a and b are the mean curvatures of E_{n+1} and E_n, respectively, then $nb = (n-1)a$.*

We now briefly consider the conditions under which the E_n de-

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* H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, Mathematische Annalen, vol. 94 (1925), pp. 123–125.

fined by (3.11) admits solutions $H(x^i, y)$ of the equations (3.7) other than $H=x^n$. By methods similar to those hitherto employed, we find that the most general solution for H of the form $H=H(x^n, y)$ is

$$H = \alpha(y)\cdot x^n + \beta(y), \qquad c = 0, \tag{3.13}$$

or

$$H = \alpha(y)\cdot x^n, \qquad c \neq 0, \tag{3.14}$$

where $\alpha(y)$ and $\beta(y)$ are arbitrary functions of y. We note that the E_{n+1} obtained by using the H defined by (3.14) coincides with (3.12).

It can be shown that solutions for H which involve some of the x^ν do not exist unless the E_n defined by (3.11) may be mapped conformally on another Einstein space in more than one way. Hence, if this is not the case, the E_n's may only be imbedded in the unique E_{n+1} defined by (3.12) if $c\neq 0$ and only in the E_{n+1}'s defined by (3.1), (3.11), and (3.13) if $c=0$. In this last case, $a=b=0$.

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CONCERNING THE BOUNDARY OF A COMPLEMENTARY DOMAIN OF A CONTINUOUS CURVE*

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Much study by various investigators has been given to the nature of the boundary of a complementary domain of a locally compact continuous curve in the plane and in certain other spaces.† It is the purpose of this paper to continue this investigation in less restricted spaces which satisfy the Jordan curve theorem and to establish certain results (from which many of the known results follow immediately) in such a way as to bring out what is essential for their validity.

It is first necessary to establish the following lemma.

LEMMA A. *If a locally compact nondegenerate continuous curve M in a complete Moore space contains no simple triod, then M is a simple continuous curve.*‡

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† See the bibliography and Chapter 4 of R. L. Moore's *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. Hereinafter, this book will be referred to as *Foundations*, and the reader is referred to it for many theorems and the definition of certain terms and phrases used in this paper.

‡ A *complete Moore space* is a space satisfying Axioms 0 and 1 of *Foundations*. A *simple continuous curve* is either a simple continuous arc, a simple closed curve, an open curve, or a ray.