$$f(x) = \int_0^\infty x(t)dp(t), \qquad x \in S. \tag{4.2}$$

Now (4.1) is a linear functional on R, and consequently a linear functional on S. Hence (4.2) states that every distributive functional on S is linear; but this is impossible unless S is finite-dimensional,* which it is not. This contradiction establishes the theorem.

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ON FUNDAMENTAL SYSTEMS OF SYMMETRIC FUNCTIONS†

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A set S of n polynomials over a field K, symmetric in n variables, $x_1, x_2, \cdots, x_n$, is said to form a fundamental system if any rational function over K, symmetric in these variables, can be expressed rationally in terms of the polynomials of S. In this paper we show that any n algebraically independent symmetric polynomials over a field K of characteristic zero form a fundamental system if the product of their degrees is less than $2n!$.

The result follows from a theorem due to Perron:‡

THEOREM 1. *Between $n+1$ polynomials (not constant), $f_1, f_2, \cdots, f_{n+1}$, in n variables, of degrees $m_1, m_2, \cdots, m_{n+1}$, respectively, there is always an identity of the form*

$$\sum C_{\nu_1\nu_2\cdots\nu_{n+1}} f_1^{\nu_1} f_2^{\nu_2} \cdots f_{n+1}^{\nu_{n+1}} \equiv 0,$$

where in each term,

$$\sum_{i=1}^{n+1} m_i\nu_i \leqq \prod_{i=1}^{n+1} m_i.$$

* Let every distributive functional on S be linear, where S is a topological vector space with the property (Q). If S is infinite dimensional, let $\{x_n\}$, $(n=1, 2, \cdots)$, be an infinite set of linearly independent elements. Since $\lim_{k\to\infty} k^{-1}x_n = \Theta$, we can choose $y_n \in S$, $(n=1, 2, \cdots)$, linearly independent, with $y_n \to \Theta$. We set $f(y_n)=1$, $f(x)=0$ when x is not a finite linear combination of the y_n, $f(ax+by)=af(x)+bf(y)$ for any $x \in S$, $y \in S$; then f is a distributive functional on S, and hence is linear on S. Since $y_n \to \Theta$, $f(y_n) \to 0$ as $n \to \infty$; but this contradicts $f(y_n)=1$. Consequently S is finite dimensional.

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‡ O. Perron, *Bemerkung zur Algebra*, Sitzungsberichte der Bayerischen Akademie, mathematisch-naturwissenschaftliche Abteilung, 1924, pp. 87–101.

The coefficients $C_{\nu_1\nu_2\cdots\nu_{n+1}}$ *belong to the coefficient field of* $f_1, f_2, \cdots, f_{n+1}$.

Consider any n algebraically independent polynomials $\phi_1, \phi_2, \cdots, \phi_n$, of degrees $m_1, m_2, \cdots, m_n$, with coefficients in a field K of characteristic zero. By Theorem 1 there exist relations

$$\Phi_i(x_i, \phi_1, \phi_2, \cdots, \phi_n) \equiv 0, \qquad i = 1, 2, \cdots, n, \tag{1}$$

each of degree less than or equal to $\prod_{i=1}^n m_i$ in x_i. The algebraic independence assures the actual presence of x_i in (1). It follows from (1) that the field $K(x_1, x_2, \cdots, x_n)$ of all rational functions of the $x_1, x_2, \cdots, x_n$ is a finite algebraic extension of the field $K(\phi_1, \phi_2, \cdots, \phi_n)$ generated by $\phi_1, \phi_2, \cdots, \phi_n$. Since K is of characteristic zero, this extension contains a primitive element ξ, which, by Theorem 1, satisfies a relation of the type (1) of degree less than or equal to $\prod_{i=1}^n m_i$ in ξ. Hence we have the following lemma:

LEMMA 1. *If* $\phi_1, \phi_2, \cdots, \phi_n$ *are n algebraically independent polynomials of degrees* $m_1, m_2, \cdots, m_n$, *then the field* $K(x_1, x_2, \cdots, x_n)$ *is a finite algebraic extension of* $K(\phi_1, \phi_2, \cdots, \phi_n)$ *of degree less than or equal to* $\prod_{i=1}^n m_i$.

The following result, which we state as a lemma, is well known:*

LEMMA 2. *If* $a_1, a_2, \cdots, a_n$ *are the elementary symmetric functions of* $x_1, x_2, \cdots, x_n$, *then* $K(x_1, x_2, \cdots, x_n)$ *is a Galois extension of* $K(a_1, a_2, \cdots, a_n)$ *of degree* $n!$.

Suppose now that $\phi_1, \phi_2, \cdots, \phi_n$ are algebraically independent and symmetric. Since $a_1, a_2, \cdots, a_n$ form a fundamental system of symmetric functions, it is clear that $K(a_1, a_2, \cdots, a_n)$ contains $K(\phi_1, \phi_2, \cdots, \phi_n)$. Hence the degree of $K(x_1, x_2, \cdots, x_n)$ over $K(\phi_1, \phi_2, \cdots, \phi_n)$ must be a multiple of the degree of $K(x_1, x_2, \cdots, x_n)$ over $K(a_1, a_2, \cdots, a_n)$. If $\prod_{i=1}^n m_i < 2n!$, it follows from Lemma 1 that the degree of $K(x_1, x_2, \cdots, x_n)$ over $K(\phi_1, \phi_2, \cdots, \phi_n)$ must be $n!$. Hence

$$K(\phi_1, \phi_2, \cdots, \phi_n) = K(a_1, a_2, \cdots, a_n),$$

and we have the theorem:

THEOREM 2. *Any set of n algebraically independent polynomials* $\phi_1, \phi_2, \cdots, \phi_n$, *symmetric in* $x_1, x_2, \cdots, x_n$, *over a field of characteristic zero forms a fundamental system if the product of their degrees is less than* $2n!$.

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* Cf. van der Waerden, *Moderne Algebra*, vol. 1, p. 173.

Theorem 2 is the best possible theorem of its kind; that is, the best general sufficiency condition for a fundamental system in terms of an upper bound for the product of the degrees without reference to the form of the polynomials $\phi_1, \phi_2, \cdots, \phi_n$. This may be verified by the example $\phi_1 = a_2$, $\phi_i = S_i$, $(i \geqq 2)$, where a_2 is the elementary symmetric function of degree 2, and S_i is the sum of the ith powers of the variables. In this case, the product of the degrees is $2n!$. The independence of $\phi_1, \phi_2, \cdots, \phi_n$ is established by showing the nonvanishing of the functional determinant D. The expression for D is

$$D = n! \cdot \begin{vmatrix} a_1 - x_1 & a_1 - x_2 & \cdots & a_1 - x_n \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

where $a_1 = x_1 + x_2 + \cdots + x_n$. After adding the second row to the first, and factoring a_1 from the first row, we have the Vandermonde determinant. Hence D does not vanish identically. On the other hand, $a_1 = (\phi_2 + 2\phi_1)^{1/2}$ is an irrational expression for a_1 whose uniqueness is guaranteed by the independence. In other words, a_1 cannot be expressed rationally in terms of the set $\phi_1, \phi_2, \cdots, \phi_n$, and the latter set does not form a fundamental system.

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