ON THE DETERMINANT OF AN AUTOMORPH OF A NONSINGULAR SKEW-SYMMETRIC MATRIX

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Let G be the skew-symmetric matrix of order $2n$,

$$G = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix},$$

where E_n is the unit matrix of order n. If F is a matrix which satisfies

$$FGF' = G, \tag{1}$$

then $|F|^2 = 1$, so that $|F| = \pm 1$. That $|F| = +1$ is well known and is in fact a consequence of a theorem of Frobenius.* A simple proof communicated to me by Professor Wintner depends on the polar factorization of F, which reduces the problem at once to the case in which F is orthogonal. This proof is, of course, not valid in any field. It is our intention here to give a simple direct proof, applicable in any field, of the fact that $|F| = +1$.

On writing F as a matrix of matrices of order n, $F = (F_{ij})$, $(i, j = 1, 2)$, we have, as a consequence of (1),

$$\begin{aligned} F_{11}F'_{12} - F_{12}F'_{11} &= F_{21}F'_{22} - F_{22}F'_{21} = 0, \\ F_{11}F'_{22} - F_{12}F'_{21} &= F_{22}F'_{11} - F_{21}F'_{12} = E_n. \end{aligned} \tag{2}$$

Let $|F_{11}| \neq 0$. Then

$$F = \begin{pmatrix} F_{11} & F_{12}F'_{11} \\ F_{21} & F_{22}F'_{11} \end{pmatrix} \begin{pmatrix} E_n & 0 \\ 0 & (F'_{11})^{-1} \end{pmatrix}.$$

On, applying (2), we have

$$\begin{aligned} |F'_{11}|\,|F| &= \begin{vmatrix} F_{11} & F_{12}F'_{11} \\ F_{21} & F_{22}F'_{11} \end{vmatrix} = \begin{vmatrix} F_{11} & F_{12}F'_{11} - F_{11}F'_{12} \\ F_{12} & F_{22}F'_{11} - F_{21}F'_{12} \end{vmatrix} \\ &= \begin{vmatrix} F_{11} & 0 \\ F_{21} & E_n \end{vmatrix} = |F_{11}|. \end{aligned}$$

* G. Frobenius, *Ueber die schiefe Invariante einer bilinearen oder quadratischen Form*, Journal für die reine und angewandte Mathematik, vol. 86 (1879), pp. 44–71; in particular, p. 48. See A. Wintner, *On linear conservative dynamical systems*, Annali di Matematica Pura ed Applicata, (4), vol. 13 (1934–1935), pp. 105–112.

Therefore $|F| = +1$, and we have proved the following lemma:

LEMMA 1. *If F_{11} is nonsingular,* $|F| = +1$.

It also follows from (2) that if $F_{12} = F_{21} = 0$, $F_{11}F'_{22} = E_n$, so that F_{11} is nonsingular and accordingly $|F| = +1$.

Let P_{ij} be the permutation matrix of order n, which by post-multiplication interchanges the ith and the jth columns of a matrix, and let P be the *diagonal block* matrix

$$P = [P_{ij}, P_{ij}] = \begin{pmatrix} P_{ij} & 0 \\ 0 & P_{ij} \end{pmatrix}.$$

Since P_{ij} is symmetric and involutory, $PGP' = G$, the matrix FP satisfies (1), and $|FP| = |F|$. Consequently we have the following lemma:

LEMMA 2. *Any matrix F_1 obtained from F by a permutation of its first n columns and the same permutation of its last n columns also satisfies* (1) *and* $|F_1| = |F|$.

The matrix $W = (W_{ij})$, $(i, j = 1, 2)$, where $W_{11} = W_{22} = [0, E_{n-1}]$ and $W_{12} = -W_{21} = [E_1, 0]$, satisfies (1) and has determinant unity. The matrix FW is obtained from F by replacing the first column by minus the $(n+1)$st column and the $(n+1)$st by the first. If, for convenience, we now write $F_{11} = A$ and $F_{12} = B$ and denote the columns of A and B by a_i and b_i, respectively, we have, as a consequence of Lemma 2, the following lemma:

LEMMA 3. *The matrix $A = F_{11}$ in F may be replaced by $C = (c_1, c_2, \cdots, c_n)$, where $c_i = a_i$ or $-b_i$.*

Therefore by Lemma 1, since $|W| = +1$, we have our fourth lemma:

LEMMA 4. *If there exists a matrix C such that $|C| \neq 0$, then* $|F| = +1$.

Let every determinant of order n formed from (A, B), in which less than r pairs of columns a_i, b_i occur with the same suffix i, be zero, but let at least one determinant with exactly r pairs of columns a_i, b_i be different from zero. As a consequence of Lemma 2 there is no loss in generality in assuming that

$$(3) \qquad |a_1 b_1 X| \neq 0,$$

where the matrix X contains exactly $r-1$ pairs of columns a_i, b_i with the same subscript i and does not contain either of the columns a_2 or b_2. Let Q be the diagonal block matrix

$$Q = \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ E_{n-1}, \ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \ E_{n-2}\right].$$

Then $|Q| = +1$, $QGQ' = G$, and FQ satisfies (1). The matrix of the first n rows of FQ is (H, K), where the n columns h_j of H are given by

$$h_j = a_j, \quad j \neq 2, \qquad h_2 = a_1 + a_2, \tag{4}$$

and the n columns k_i of K by

$$k_i = b_i, \quad i \neq 1, \qquad k_1 = b_1 - b_2. \tag{5}$$

Since the matrix X in (3) does not contain any of the columns a_1, b_1, a_2, b_2, it follows from (4) and (5) that the matrix $T = (h_2 k_1 X)$ is a submatrix of (H, K), which contains exactly $r-1$ pairs of columns h_i, k_i with the same suffix i, and that

$$\begin{aligned} |T| &= |a_1 + a_2, b_1 - b_2, X| \\ &= |a_1 b_1 X| - |a_1 b_2 X| + |a_2 b_1 X| - |a_2 b_2 X|. \end{aligned} \tag{6}$$

But, by hypothesis,

$$|a_1 b_2 X| = |a_2 b_1 X| = 0. \tag{7}$$

Since $|a_1 a_2 X|$ is also zero and $|a_1 b_1 X|$ is not zero by (3), a_2 and b_2 are both linear combinations of the $n-1$ columns of the matrix $(a_1 X)$. Hence $|a_2 b_2 X| = 0$ and, as a consequence of (6) and (7), $|T| = |a_1 b_1 X| \neq 0$.

Therefore in (H, K) there is one nonzero subdeterminant of order n which contains exactly $r-1$ pairs of columns h_i, k_i with the same suffix i.

Now $|FQ| = |F|$, and FQ satisfies (1). Further, the matrix C in Lemma 4 contains exactly $r = 0$ pairs of columns a_i, b_i. By a simple induction proof we therefore have the following lemma:

LEMMA 5. *If in (A, B) there is one nonzero subdeterminant of order n which contains $r \leqq n$ pairs of columns a_i, b_i with the same suffix i, then* $|F| = +1$.

Since any matrix F which satisfies (1) is nonsingular, the rank of (A, B) is n and Lemma 5 implies the following statement:

THEOREM. *If $FGF' = G$,* $|F| = +1$.

This proof is valid in any field.

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