SOME INVARIANTS UNDER MONOTONE TRANSFORMATIONS*

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We assume that S is a locally connected, connected, compact metric space and that P is a property of point sets. For any two points a and b of S we denote by $C(ab)$ (respectively $C_i(ab)$) a closed (closed irreducible) cutting of S between the points a and b. We consider the following properties:

$\Delta_0(P)$. *If S is the sum of two continua, their product has property P.*

$\Delta_1(P)$. *If K is a subcontinuum of S and R is a component of $S-K$, then the boundary of R, $(F(R)=\overline{R}-R)$, has property P.*

$\Delta_2(P)$. *Each $C_i(ab)$ has property P.*

$\Delta_3(P)$. *If A and B are disjoint closed sets containing the points a and b, respectively, there is a $C(ab)$ disjoint from $A+B$ and having property P.*

If P is the property of being connected, the four properties $\Delta_i(P)$ are equivalent as shown by Kuratowski.‡ Indeed it may be seen that Kuratowski's proofs allow us to state the following theorem:

THEOREM 1. *For any property P of point sets, $\Delta_i(P)$ implies $\Delta_{i+1}(P)$ for $i=0, 1, 2$.*

This result is the best possible in the sense that there is a property (that of being totally disconnected) for which no other implication holds.

The single-valued continuous transformation $T(S)=S'$ is said to be *monotone* if the inverse of every point is connected. It may be seen that the following statements are true:§

(i) *The inverse of every connected set is connected.*

(ii) *If the set X separates S between the inverses of the points x and y, then $T(X)$ separates S' between x and y.*

THEOREM 2. *If the property P is invariant under monotone trans-*

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* Presented to the Society, October 29, 1938.

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‡ C. Kuratowski, *Une caractérisation topologique de la surface de la sphère*, Fundamenta Mathematicae, vol. 13 (1929), p. 307, and references given there.

§ G. T. Whyburn, *Non-alternating transformations*, American Journal of Mathematics, vol. 56 (1934), p. 294.

formations, then for each $i=0, 1, 2, 3$, the property $\Delta_i(P)$ is invariant under the monotone transformation $T(S)=S'$.

PROOF. (0) If $S'=L+M$, the summands being continua, then $S=L^{-1}+M^{-1}$* is a sum of continua. Hence the set $L^{-1}\cdot M^{-1}$ has property P and $L\cdot M=T(L^{-1}\cdot M^{-1})$ then has property P.

(1) If R is a component of $S'-K$, where K is a continuum, then R^{-1} is a component of the complement of the continuum K^{-1}. By assumption, $F(R^{-1})$ has property P. It follows that its image has property P. But we have $T(F(R^{-1}))=T(\overline{R^{-1}}-R^{-1})=\overline{T(R^{-1})}-R=F(R)$.

(2) Assume that C is a $C_i(ab)$ in S'. From the continuity of T it follows that C^{-1} is a $C(pq)$ in S, p and q being any two points in the inverses of a and b, respectively. Since the inverses of a and b are connected, there exists a cutting K of S† between these two sets such that K is a $C_i(xy)$, where $T(x)=a$ and $T(y)=b$; and further K is a subset of C^{-1}. Thus K has property P; hence $T(K)$ has. But $T(K)\subset C$, and $T(K)$ is a $C(ab)$. It follows that $T(K)=C$ and from this that C has property P.

(3) Let A and B denote disjoint closed subsets of S' containing a and b. If x and y are points which map into a and b, then by hypothesis there is a cutting K of S between x and y that is disjoint with A^{-1} and B^{-1} and has property P. Since, clearly, K is a cutting of S between the inverses of a and b, it follows that $T(K)$ cuts S' between a and b, is disjoint with $A+B$, and has property P.

As an application we have the following known results:‡

THEOREM 3. *The property of a locally connected continuum to be a dendrite, a regular curve, or a rational curve is a monotone invariant.*

To see this we take P to be the property of being a point, a finite set of points, or a countable set of points and apply the invariance of $\Delta_3(P)$.

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* If X is a subset of S', we denote by X^{-1} the inverse of X.

† G. T. Whyburn, *Concerning irreducible cuttings of continua*, Fundamenta Mathematicae, vol. 13 (1929), p. 42.

‡ See the fourth footnote and references given there.