NEVANLINNA ON ANALYTIC FUNCTIONS

Eindeutige analytische Funktionen. By R. Nevanlinna. (Grundlehren der mathematischen Wissenschaften, vol. 46.) Berlin, Springer, 1936. 6+353 pp.

The list of contents is as follows: Introduction. 1. Conformal mapping of simply or multiply connected domains. 2. Solution of Dirichlet's problem for a smooth domain. 3. The principle of the harmonic measure and its applications. 4. Relations between euclidean and non-euclidean determinations of measure. 5. Point sets of harmonic measure zero. 6. The first fundamental theorem in the theory of meromorphic functions. 7. Functions of bounded type. 8. Meromorphic functions of finite order. 9. The second fundamental theorem in the theory of meromorphic functions. 10. Applications of the second fundamental theorem. 11. The Riemann surface of univalent functions. 12. The type of a Riemann surface. 13. Ahlfors' theory of covering surfaces. Literature. Index.

The theory of functions of a complex variable is largely the creation of the nineteenth century. At the turn of the century the attention of the analysts was diverted into other channels. Integral equations and especially the theory of integration were new fields which promised heavy returns and attracted most of the budding analysts. The Scandinavians, however, with their usual ability to absorb and develop new ideas without sacrificing what is valuable in the old ones, remained faithful to their older allegiance. In Helsingfors, in particular, the able leadership of E. Lindelöf created a school of analysts which is nowadays leading in the field of complex function theory. R. Nevanlinna is one of the most brilliant exponents of this school, and the present treatise gives an account essentially of their work in which he has taken such a prominent part.

The central question in the work of this school is the value distribution problem for functions meromorphic in a domain. The geometrical aspect of this problem is the question of the structure of certain classes of Riemann surfaces. As basic tools in the study of the problem figure the theory of harmonic functions and various non-euclidean notions of measure, in particular, harmonic and hyperbolic measures. All material in the book is grouped around these ideas and their interrelations. It is impossible for me to give an adequate account of this excellent book within reasonable compass, but some remarks concerning the main questions will perhaps induce some of my readers to read the book itself.

Let us start with the value distribution problem. It is supposed that $w=f(z)$ is single-valued and analytic save for poles in the domain D: $|z|<R\leqq\infty$. What can be said concerning the distribution of the values of this function in D? What values are actually taken on, which values are omitted, and which are approached as the variable z tends towards the boundary of D? Let us represent the values of w by stereographic projection upon a sphere of radius 1/2. Then $w=f(z)$ maps the region $|z|\leqq r<R$ on a Riemann surface F_r on the sphere. Let the area of F_r be $\pi A(r)$. Then $A(r)$ measures the number of times that F_r covers the sphere and may be called the average number of sheets of F_r; it is intimately connected with the value distribution problem. Let $n(r, a)$ be the number of roots of the equation $f(z)=a$ for $|z|\leqq r$, each root being counted with its proper multiplicity. If $A(r)\to\infty$ as $r\to R$, one might be led to suspect that $n(r, a)\sim A(r)$ for almost all a. This is true for sufficiently simple functions $f(z)$, but in general the functions $A(r)$ and $n(r, a)$ are too irregular for the validity of such asymptotic relations. This difficulty can be eliminated, however, by a

suitable integration process which smoothes out the irregularities. This process leads to the interesting equality

$$T(r) \equiv \int_0^r A(t)\frac{dt}{t} = N(r, a) + m(r, a). \tag{1}$$

This is Nevanlinna's first fundamental theorem in the form due to Shimizu and Ahlfors. Here $T(r)$ is his characteristic function, and

$$N(r, a) = \int_0^r [n(t, a) - n(0, a)]\frac{dt}{t} + n(0, a)\log r, \tag{2}$$

$$m(r, a) = \frac{1}{2\pi}\int_0^{2\pi} \log \chi[f(re^{i\theta}), a]d\theta - \log \chi[f(0), a], \tag{3}$$

where $\chi[w, a]$ is the reciprocal of the chordal distance between the points w and a on the sphere. The enumerative function $N(r, a)$ measures the number of times that $f(z)$ actually becomes equal to a for $|z| \leqq r$, whereas the osculating function (Schmiegungsfunktion) $m(r, a)$ measures the rate of approach of $f(z)$ to the value a as $|z| \to R$. Equation (1) states that $f(z)$ has the same affinity to all values if properly measured. This statement becomes somewhat illusory if $T(r)$ remains bounded. This situation arises if and only if $R < \infty$ and $f(z)$ is the quotient of two functions bounded in $|z| < R$. Nevanlinna says that such a function is of bounded type (beschränktartig).

Suppose now that $T(r) \to \infty$ as $r \to R$. Since the integral of $m(r, a)$ over any measurable set of a-values on the sphere stays bounded as $r \to R$, we conclude that $T(r) \sim N(r, a)$ for almost all values of a. Thus the frequency is approximately the same for almost all a. This result replaces the hypothetical relation $A(r) \sim n(r, a)$ which it yields upon formal differentiation.

More precise information is given by the second fundamental theorem, which takes the form of the inequality

$$\sum_{\nu=1}^{q} m(r, a_\nu) < 2T(r) - N_1(r) + S(r). \tag{4}$$

Here $q > 2$ and $N_1(r)$ is an enumerative function of the same type as $N(r, a)$, obtained by replacing $n(r, a)$ in (2) by $n_1(r)$, the number of multiple values of $f(z)$ in $|z| \leqq r$, each k-tuple value being counted $k-1$ times only. Finally $S(r)$ is a remainder which usually is $O(\log r)$ if $R = \infty$ and $O\{|\log (R-r)|\}$ if $R < \infty$.

Formula (4) leads to the defect and ramification relations. Let $N_1(r, a)$ be obtained by replacing $n(r, a)$ in (2) by $n_1(r, a)$, the number of multiple roots of the equation $f(z) = a$, each k-fold root being counted $k-1$ times, and put $\overline{N}(r, a) = N(r, a) - N_1(r, a)$. Nevanlinna makes the following definitions:

$$\delta(a) = \liminf_{r\to R} \frac{m(r, a)}{T(r)}, \quad \theta(a) = \liminf_{r\to R} \frac{N_1(r, a)}{T(r)}, \quad \Theta(a) = 1 - \limsup_{r\to R} \frac{\overline{N}(r, a)}{T(r)}, \tag{5}$$

where $\delta(a)$ is known as the defect of a, while $\theta(a)$ and $\Theta(a)$ are known as the algebraic and the total ramification index, respectively. For every a, $\delta(a) + \theta(a) \leqq \Theta(a) \leqq 1$. Formula (4) now gives the fundamental relations*

$$\sum \delta(a) + \sum \theta(a) \leqq \sum \Theta(a) \leqq 2, \tag{6}$$

where the summation extends over all values of a. The formula shows that $\delta(a)$ and $\theta(a)$ can be different from zero for at most a denumerable set of values a.

The formula contains a number of important special theorems. The defect $\delta(a)$ has its maximal value 1 if $N(r, a) = o[T(r)]$, in particular, if $f(z) \neq a$ for all z. Formula

* If $R < \infty$, it is supposed that $\log (R-r)/T(r) \to 0$ when $r \to R$.

(6) shows that there are at most two such values unless $f(z)$ is a constant. This is Picard's theorem. A value a is said to be completely ramified if the equation $f(z)=a$ has only multiple roots. For such a value $\Theta(a)\geqq 1/2$, so there can be at most four such values. For the Weierstrass $\wp$-function the values e_1, e_2, e_3, and ∞ are completely ramified, so the theorem is the best possible of its kind. This theorem also leads to a new proof of another classical theorem due to Picard, according to which an algebraic relation of genus greater than one cannot be uniformized by functions meromorphic for $|z|<\infty$.

Nevanlinna's fundamental theorems are metric statements concerning the structure of certain Riemann surfaces. The defect relation raises the problem of the existence of meromorphic functions having preassigned defects at preassigned points. If the points are finite in number and the defects are taken as rational numbers having the sum 2, the problem has been completely solved by Nevanlinna. It is enough to construct a Riemann surface having a suitable number of logarithmic branch points over each of the preassigned points. The corresponding mapping function $w=f(z)$ is characterized by the fact that its Schwarzian derivative is a polynomial in z, and vice versa.

It is known that a simply connected Riemann surface can be mapped in a one-to-one and conformal (except for possible interior branch points of finite order) manner upon one of the following domains: (i) the whole plane, (ii) the punctured plane, (iii) the interior of the unit circle. The surface is said to be of the elliptic, parabolic, or hyperbolic type according as (i), (ii), or (iii) holds. The elliptic case is elementary, but it is not an easy matter to tell from a description of the structure of the surface whether the parabolic or the hyperbolic case holds. The frequency of the branch points seems to be the decisive factor. The reader will find some results in this connection in Chapter 12 of the treatise under review.

The geometric side of the theory has been put upon a very general basis through the brilliant investigations of Ahlfors on covering surfaces. On the basic surface F_0, which may be closed or have a boundary, he supposes the existence of a finite triangulation and of a metric satisfying certain mild restrictions. A covering surface F^* of F_0 having infinitely many sheets is supposed to be generated by a process of regular exhaustion from a sequence of finite covering surfaces F_k. The boundary of F_k relative to F_0 is taken to be rectifiable in the metric and of length L_k. If S_k is the average number of sheets of F_k, that is, the ratio between the area of F_k and that of F_0, then L_k/S_k should tend to zero with $1/k$. Let D be an arbitrary domain on F_0. That part of F_k which lies above D consists of a certain number of connected pieces which are of two types. Those which are unbounded relative to D are referred to as islands, whereas the remaining pieces will be called peninsulas (Zungen, in Nevanlinna's terminology). Let $S_k(D)$ be the average number of sheets of F_k above D, let $n_k(D)$ be the number of sheets which belong to islands above D, and put $S_k(D)=n_k(D)+m_k(D)$. Then the first fundamental theorem has the following analogue. There exists a constant $h(D)$, depending upon D only, such that

$$n_k(D)+m_k(D)=S_k+\eta_kL_k,\qquad |\eta_k|<h(D),\tag{7}$$

for every k.†

For the second fundamental theorem we suppose in addition that F_0 is a closed surface of genus zero, which we can take to be the sphere, and that the open covering surface F^* is simply connected. Consider $q>2$ disjoint domains D_j on F_0. Let the simple multiplicity of an island be defined as the negative of its Euler characteristic, and let $p_k(D_j)$ be the sum of the simple multiplicities of the islands of F_k above D_j.

† Formula (A) on p. 332 seems to be faulty.

Then there exists a constant h, depending only upon $D_1, \cdots, D_q$, such that

$$(q-2)S_k - hL_k \leqq \sum_{j=1}^{q} p_k(D_j) \leqq qS_k + hL_k. \tag{8}$$

This inequality corresponds to the second fundamental theorem and gives rise to corresponding defect and ramification relations. Let $n_{k,1}(D)$ denote the sum of the orders of the branch points of F_k above D, and put

$$\delta(D) = \liminf_{k\to\infty} \frac{m_k(D)}{S_k}, \qquad \theta(D) = \liminf_{k\to\infty} \frac{n_{k,1}(D)}{S_k}. \tag{9}$$

These are the defect and the ramification index respectively of D with respect to the covering surface F^*. We have the basic relation

$$\sum_{j=1}^{q} \left[\delta(D_j) + \theta(D_j)\right] \leqq 2. \tag{10}$$

From this relation we can draw conclusions which are obvious analogues of the theorems quoted above. Thus, for instance, a regularly exhaustible, simply connected covering surface of the sphere covers every point of the sphere with at most two exceptions. There can be at most four disjoint domains in each of which every sheet of F^* has a branch point. These theorems apply in particular to the case in which F^* is the conformal map of the interior of the circle $|z| < R \leqq \infty$ by a meromorphic function $w=f(z)$. If $R=\infty$ the surface is always regularly exhaustible, and if $R<\infty$ it is sufficient that lim sup $(R-r)\ A(r)=\infty$. The resulting theorems complete those of Nevanlinna in an interesting manner.

It remains to say a few words about the basic notion in the first part of the book, that of harmonic measure. Let D be a domain in the complex plane, B its boundary, and $B=\alpha+\beta$ a disjunction of B. For sufficiently simple domains D and boundary sets α there exists a uniquely determined function $\omega(z, \alpha, D)$ which is bounded and harmonic in D and takes on the value 1 on α and 0 on β. This function is referred to as the harmonic measure of the set α with respect to the domain D at the point z. If D is mapped conformally in a one-to-one manner upon D^* so that z goes into z^* and α into α^*, then $\omega(z, \alpha, D)=\omega(z^*, \alpha^*, D^*)$. But if the correspondence is merely analytic without being one-to-one, the harmonic measure is increased, or at least never decreased. This principle of the harmonic measure turns out to be a very fertile source of important inequalities. Another such source is Carleman's principle of extension: if D is extended across β, then the harmonic measure increases. From these principles the author derives the well known theorems of Phragmén-Lindelöf, Landau, and Schottky, further, the deformation theorems of Koebe and of Ahlfors, to mention only a few examples.

In closing this review permit me to point out some problems in modern function theory to which the Helsingfors school has made important contributions, but which have been omitted in Nevanlinna's book. One would have liked to have seen a discussion of functions meromorphic in other domains than the interior of a circle or the punctured plane. The case of a half plane occurs often in practice. While this case can be reduced to that of a circle by conformal mapping, this method does not yield the most convenient formulas. The original memoirs of F. Nevanlinna and R. Nevanlinna being not easily accessible, the inclusion of this material would have been of great service to the mathematical public. One also misses a discussion of theorems of the Blaschke or Carlson types with their many generalizations. A synopsis of the present day stand of this question would have found its natural place in the book under review.

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