ORTHOGONAL POLYNOMIALS WITH ORTHOGONAL DERIVATIVES*

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1. **Introduction.** Let $\{\phi_n(x) \equiv x^n + \cdots\}$ be a set of orthogonal polynomials satisfying the relations

$$
(1) \quad \begin{aligned}
&\int_a^b p(x)\phi_m(x)\phi_n(x)dx = \int_a^b q(x)\phi_m'(x)\phi_n'(x)dx = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad m \neq n;\, m, n = 0, 1, \cdots, \\
&\alpha_i \equiv \int_a^b p(x)x^i dx, \qquad \beta_i \equiv \int_a^b q(x)x^i dx, \qquad i = 0, 1, \cdots, \\
&p(x) \geqq 0, \qquad q(x) \geqq 0, \qquad \alpha_0 > 0, \qquad \beta_0 > 0.
\end{aligned}
$$

Lebesgue integrals are used and the interval (a, b)† may be finite or infinite.

We are concerned with the following assertion:

THEOREM. *If $\{\phi_n(x)\}$ and $\{\phi_n'(x)\}$ are orthogonal systems of polynomials, then $\{\phi_n(x)\}$ may be reduced to the classical polynomials of Jacobi, Laguerre, or Hermite by means of a linear transformation on x.*

This result was first proved by W. Hahn‡ who obtained a differential equation of the second order for $\phi_n(x)$. When (a, b) is finite, Krall§ derived the Jacobi polynomials by using the moments β_i to determine the weight function $q(x)$. The present paper extends his method to the case (a, b) infinite, thus obtaining the Laguerre and Hermite polynomials.

2. **Weight function for $\{\phi_n'(x)\}$.** Krall's proof shows that constants r, s, t (not all zero) may be determined so that

* Presented to the Society, November 28, 1936.

† There is no loss of generality in assuming the intervals of orthogonality for $\{\phi_n(x)\}$ and for $\{\phi_n'(x)\}$ to be the same, since the definitions of $p(x)$, $q(x)$ may always be extended to a common interval (a, b). More generally, $p(x)dx$ may be replaced by $d\psi_1(x) \equiv Ap(x) + dT(x)$, where A is a constant, and $\int_a^b x^i dT(x) = 0$, $(i = 0, 1, \cdots)$; $q(x)dx$ may be replaced by $d\psi_2(x)$, where $\psi_2(x)$ is monotone non-decreasing.

‡ W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Mathematische Zeitschrift, vol. 39 (1935), pp. 634–638.

§ H. Krall, *On derivatives of orthogonal polynomials*, this Bulletin, vol. 42 (1936), pp. 423–428.

$$(2)\qquad \int_a^b q_1(x)x^i dx = \int_a^b q(x)x^i dx, \qquad q_1(x) \equiv (rx^2 + sx + t)p(x), \quad i = 0, 1, \cdots .$$

We suppose that (a, b) is the smallest interval in the sense that no number h, $(a<h<b)$, exists such that $\int_a^h p(x)dx=0$ or $\int_h^b p(x)dx=0$. There is no restriction in assuming likewise that either $(a, b)=(0, \infty)$ or $(a, b)=(-\infty, \infty)$. (Perform, if necessary, a linear transformation on x.)

Following Krall we let

$$(3)\qquad S(x) = K\int_a^x (z - L)p(z)dz, \qquad a \leqq x \leqq b,$$

where K, L are constants determined by the conditions $S(b)=0$, $\int_a^b S(x)dx=\int_a^b q(x)dx$. The boundary conditions on $S(x)$ require that the integrand $(z-L)p(z)$ change sign so that $a<L<b$. Then $\int_a^x (z-L)p(z)dz$ decreases in (a, L) and increases in (L, b), therefore this integral is always less than or equal to zero. Hence,

$$\int_a^b K\left(\int_a^x (z - L)p(z)dz\right)dx = \int_a^b q(x)dx > 0$$

requires $K<0$ and therefore $S(x)\geqq 0$. Suppose

$$\epsilon_x \equiv \int_x^\infty K(z - L)p(z)dz$$

and

$$\epsilon_x' \equiv \int_x^\infty Kz^i(z - L)p(z)dz,$$

i a positive integer. Then, $S(x)=-\epsilon_x$, $-\epsilon_x'\geqq -\epsilon_x x^i$ if $x>|L|$, and ϵ_x, $\epsilon_x'\to 0$ as $x\to\infty$. Therefore, $S(x)\leqq -\epsilon_x'/x^i$ if $x>|L|$, and $x^iS(x)\to 0$ as $x\to\infty$, $(i=0, 1, \cdots)$. Similarly, if $a=-\infty$, we prove that $x^iS(x)\to 0$ as $x\to-\infty$, $(i=0, 1, \cdots)$. In every case, $\int_a^b x^iS(x)dx$ exists, $(i=0, 1, \cdots)$. We conclude that $S(x)$ has the following properties:

$$(4)\qquad \begin{aligned} &K < 0,\ a < L < b,\ S(x) > 0,\ a < x < b,\ S(a) = S(b) = 0,\\ &S'(x) = K(x - L)p(x) \text{ exists almost everywhere},\\ &S'(x) \geqq 0,\ a \leqq x \leqq L, \text{ almost everywhere},\\ &S'(x) \leqq 0,\ L \leqq x \leqq b, \text{ almost everywhere},\\ &x^iS(x) \to 0 \text{ as } x \to a \text{ or } b, \qquad i = 0, 1, \cdots . \end{aligned}$$

Again, with Krall, we obtain

$$\int_a^b S(x)x^i dx = \int_a^b q_1(x)x^i dx = \int_a^b q(x)x^i dx, \qquad i = 0, 1, \cdots,$$

$$(5) \qquad \int_a^b S(x)\phi_m'(x)\phi_n'(x)dx = \int_a^b q_1(x)\phi_m'(x)\phi_n'(x)dx = 0,$$

$$m \neq n; m, n = 0, 1, \cdots,$$

$$q_1(x) \equiv S(x) + T(x), \quad \int_a^b T(x)x^i dx = 0, \qquad i = 0, 1, \cdots.$$

In the finite interval this requires that $T(x) \equiv 0$ almost everywhere, but in the infinite interval this result does not follow.* However, it is known that if $\int_{-\infty}^{\infty} T(x)x^i dx = 0$, $(i = 0, 1, \cdots)$, and if $\int_{-x}^{x} |T(z)| dz$ exists for every x, and $T(x) \geqq 0$ for $|x|$ sufficiently large, then $T(x) \equiv 0$ almost everywhere. We shall now prove this statement.

Suppose $T(x) \geqq 0$ for $|x| \geqq A$. In view of (5), $\int_{-x}^{x} |T(z)| dz$ exists for all x. Choose $A' > A$ and i even. Then

$$\int_{-\infty}^{\infty} T(x)x^i dx = \int_{-\infty}^{-A} T(x)x^i dx + \int_{-A}^{A} T(x)x^i dx + \int_{A}^{A'} T(x)x^i dx$$

$$+ \int_{A'}^{A'+1} T(x)x^i dx + \int_{A'+1}^{\infty} T(x)x^i dx = \sum_{n=1}^{5} I_n = 0,$$

where $I_1 \geqq 0$, $I_3 \geqq 0$, $I_4 \geqq 0$, $I_5 \geqq 0$, $I_2 \leqq 0$, $I_1 + I_3 + I_5 \geqq 0$, and $I_2 + I_4 \leqq 0$. Given $\epsilon > 0$, suppose $T(x) \geqq \epsilon$ on some set G of positive measure in (A, ∞). Choose A', $(A' > A)$, such that the interval $(A', A'+1)$ contains a subset of G of measure $\sigma > 0$. Then $I_4 > 0$ and

$$\frac{|I_2|}{I_4} \leqq \frac{\int_{-A}^{A} x^i \cdot |T(x)| dx}{\sigma \cdot \epsilon (A')^i} \leqq \frac{A^i \int_{-A}^{A} |T(x)| dx}{\sigma \cdot \epsilon (A')^i} < 1$$

if i is sufficiently large, since $A/A' < 1$. Then $|I_2| < I_4$, $I_2 + I_4 > 0$, which is a contradiction. Thus $T(x) \equiv 0$ almost everywhere in (A, ∞), and likewise in $(-\infty, -A)$. We conclude that $\int_{-A}^{A} T(x)x^i dx = 0$, $(i = 0, 1, \cdots)$, therefore $T(x) \equiv 0$ almost everywhere.

Since $S'(x) = K(x-L)p(x)$ almost everywhere, (2) and (5) lead to the differential equation

$$(6) \qquad (rx^2 + sx + t)S'(x) - K(x - L)S(x) = K(x - L)T(x).$$

The solution of (6) is

* Stieltjes' example is $\int_0^\infty x^n e^{-x^{1/4}} \sin(x^{1/4})dx = 0$, $(n = 0, 1, \cdots)$.

$$
(7)\qquad \begin{aligned} S(x) &= S_1(x) + Cf(x), \qquad \log f(x) = \int^x \frac{K(z-L)dz}{rz^2+sz+t}, \\ S_1(x) &= f(x)\int_c^x \frac{K(z-L)T(z)dz}{(rz^2+sz+t)f(z)}, \qquad c,\ C \text{ constants.} \end{aligned}
$$

3. **Discussion of** rx^2+sx+t. (i) Suppose first that rx^2+sx+t has imaginary zeros. Then

$$
f(x) = (rx^2+sx+t)^\alpha e^{\beta \arctan(\gamma x+\delta)}, \qquad \alpha,\ \beta,\ \gamma,\ \delta \text{ constants,}
$$

where $r[1+(\gamma x+\delta)^2] \equiv \gamma^2(rx^2+sx+t)$, $2\alpha r = K$, and $\beta r = -\alpha(2rL+s)$. Since $\beta_0 = \int_a^b q_1(x)dx > 0$, we conclude that $r>0$, $\alpha<0$, and $rx^2+sx+t > 0$ in (a, b).

Let i be an integer such that $\alpha+i \geqq 0$, $i\geqq 1$, and let $f_1(x) \equiv (rx^2+sx+t)^i [(\alpha+i)(2rx+s)+\beta r/\gamma]f(x)$.

Integrating by parts we have

$$
\int_a^b \frac{S(x)}{f(x)} f_1'(x)dx = \left[f_1(x)\left\{\int_c^x \frac{K(z-L)T(z)dz}{(rz^2+sz+t)f(z)} + C\right\}\right]_a^b
$$

$$
- K\int_a^b (x-L)(rx^2+sx+t)^{i-1}\left[(\alpha+i)(2rx+s)+\frac{\beta r}{\gamma}\right]T(x)dx,
$$

$$
\int_a^b S(x)(rx^2+sx+t)^{i-1}\left\{\left[(\alpha+i)(2rx+s)+\frac{\beta r}{\gamma}\right]^2 + 2r(\alpha+i)(rx^2+sx+t)\right\}dx = 0.
$$

Since the integrand does not change sign, we conclude that $S(x)\equiv 0$ almost everywhere, which is impossible in view of (4).

(ii) Suppose that $rx^2+sx+t = r(x-g)^2$. As in (i), $r>0$. Here

$$
f(x) = (x-g)^\alpha e^{\beta/(x-g)}, \qquad \alpha,\ \beta \text{ constants.}
$$

Let i be an integer such that $\alpha+i\geqq 0$, $i\geqq 2$, and

$$
f_1(x) \equiv (x-g)^i[(\alpha+i)(x-g)-\beta]f(x).
$$

As in (i),

$$
\int_a^b \frac{S(x)}{f(x)} f_1'(x)dx = 0,
$$

which is impossible.

(iii) Suppose that $rx^2+sx+t = r(x-g)(x-h)$, (g, h real; $g<h$). (If (a, b) is finite, this is the only possible case, since $T(x)\equiv 0$ almost everywhere.) Here $f(x) = (x-g)^\alpha(x-h)^\beta$,

$$S(x) = (x-g)^{\alpha}(x-h)^{\beta}\left\{\int_c^x \frac{K(z-L)T(z)dz}{r(z-g)^{\alpha+1}(z-h)^{\beta+1}} + C\right\},$$

$$\alpha,\ \beta,\ c,\ C \text{ constants, } r(\alpha+\beta) = K.$$

If i, j are integers such that $\alpha+i>0$, $\beta+j>0$, $i\geqq 1$, $j\geqq 1$, then, integrating by parts, we find that

$$\int_a^b \frac{S(x)}{(x-g)^{\alpha}(x-h)^{\beta}} \frac{d}{dx}\left[(x-g)^{\alpha+i}(x-h)^{\beta+j}\right]dx = 0,$$

$$\int_a^b S(x)(x-g)^{i-1}(x-h)^{j-1}(x-\bar{x})dx = 0,\ \bar{x} = \frac{(\alpha+i)h+(\beta+j)g}{\alpha+\beta+i+j}.$$

This is impossible, as we shall show when $a=0$. (The proof is similar if $a=-\infty$.) If $\bar{S}(x)\equiv S(x)(x-g)^{i-1}(x-h)^{j-1}(x-\bar{x})$, and if constants A, A' are chosen so that $A'>A>3|h|+|L|+1$, then

$$\int_0^{\infty}\bar{S}(x)dx = \int_0^{A}\bar{S}(x)dx + \int_A^{A'}\bar{S}(x)dx + \int_{A'}^{A'+1}\bar{S}(x)dx$$

$$+\int_{A'+1}^{\infty}\bar{S}(x)dx \equiv \sum_{n=1}^{4} i_n = 0.$$

If i is so large that $|\bar{x}-h|<|h|+1$, we have

$$i_2>0,\quad i_3>0,\quad i_4>0,\quad i_1<0,\quad i_2+i_4>0,\quad i_1+i_3<0.$$

On the other hand,

$$\frac{|i_1|}{i_3} \leqq \frac{(A-g)^{i-1}(A-h)^{j-1}(A-\bar{x})}{(A'-g)^{i-1}(A'-h)^{j-1}(A'-\bar{x})S(A'+1)}\int_0^A S(x)dx < 1$$

if i is sufficiently large. Then $|i_1|<i_3$, contradicting $i_1+i_3<0$.

From these cases we conclude that $r=0$.

(iv) Suppose that $r=0$, $s\neq 0$. Let

$$\frac{K(x-L)}{sx+t} = \beta + \frac{\alpha s}{sx+t}, \qquad \alpha,\ \beta \text{ constants, } \beta s = K.$$

The condition $S(b)=0$ gives

$$\int_a^b \left[\beta(sx+t)p(x)+\alpha s p(x)\right]dx = \beta\int_a^b q_1(x)dx + \alpha s\int_a^b p(x)dx$$

$$= \beta\beta_0 + \alpha s\alpha_0 = 0.$$

We must have $\alpha>0$, because α_0, $\beta_0>0$, $\beta s<0$. In this case, $f(x)=(sx+t)^{\alpha}e^{\beta x}$, and

$$S(x) = (sx+t)^{\alpha}e^{\beta x}\left\{\int_c^x \frac{K(z-L)T(z)dz}{(sz+t)^{\alpha+1}e^{\beta z}} + C\right\}, \qquad c, C \text{ constants.}$$

If $a<-t/s\equiv -t'$, the existence of $S(x)$ near $-t'$ requires the existence of the integral

$$\int_c^{-t'} K(z-L)(sz+t)^{-\alpha-1}e^{-\beta z}T(z)dz,$$

so that $S(-t')=0$, which is impossible. Thus $sx+t$ does not change sign in (a, b), and $s>0$ because $\beta_0>0$. Then $a=0$, $\beta<0$, $t'\equiv t/s\geqq 0$. Let

$$S_2(x) \equiv \begin{cases} 0 & \text{in} \quad (0, t'), \\ S(x-t') & \text{in} \quad (t', \infty), \end{cases}$$

where

$$S(x-t') \equiv \frac{1}{s}\, x^{\alpha}e^{\beta x}\left\{\int_{c'}^x \frac{K(z-L-t')T(z-t')dz}{z^{\alpha+1}e^{\beta z}} + C'\right\},$$

c', C' constants.

The weight functions $S(x-t')$ in (t', ∞) and $S_2(x)$ in $(0, \infty)$ give rise to the same system of orthogonal polynomials $\{\phi_n'(x-t')\}$ since the moments are the same. Let

$$T_1(x) \equiv S_2(x) + C_1x^{\alpha}e^{\beta x},$$

where the constant C_1 is determined so that $\int_0^\infty T_1(x)dx=0$. Integrating by parts, we obtain

$$\int_0^\infty \frac{T_1(x)}{x^{\alpha}e^{\beta x}} \frac{d}{dx}\left[x^{\alpha+i}e^{\beta x}\right]dx = 0, \qquad i \geqq 1,$$

$$\int_0^\infty T_1(x)x^{i-1}[\alpha+i+\beta x]dx = \int_0^\infty T_1(x)x^{i-1}dx = 0, \qquad i = 1, 2, \cdots.$$

Hence, if we neglect the function $T_1(x)$ whose moments vanish, the weight function is of the form $Cx^{\alpha}e^{\beta x}$ (C an arbitrary constant). Replacing x by $-x/\beta$ and putting $C=(-\beta)^{\alpha}$, we obtain the weight function $x^{\alpha}e^{-x}$ which is the weight function for the derivatives of the Laguerre polynomials with the property that

$$\int_0^\infty x^{\alpha-1}e^{-x}\phi_m(x)\phi_n(x)dx = \int_0^\infty x^\alpha e^{-x}\phi_m'(x)\phi_n'(x)dx = 0,$$

$$\alpha > 0;\ m \neq n;\ m,\ n = 0,\ 1,\ \cdots.$$

(v) Suppose that $r=s=0$, $t\neq 0$. Since $\beta_0>0$, we must have $t>0$. Here $f(x)=e^{K'(x^2-2Lx)}$, and

$$S(x) = e^{K'(x^2-2Lx)}\left\{\int_c^x 2K'(z-L)e^{-K'(z^2-2Lz)}T(z)dz + C\right\},$$

$$K',\ c,\ C \text{ constants},\ K' = K/2t < 0.$$

If i is an odd positive integer, the function

$$e^{-K'(x^2-2Lx)}\int_\infty^x (z-L)^i e^{K'(z^2-2Lz)}dz$$

is a polynomial in x. Hence, integration by parts gives $\int_a^b S(x)(x-L)^i dx =0$, ($i$ odd), which requires $a=-\infty$. Since by (5)

$$\int_{-\infty}^\infty p(x)(x-L)^i dx = \int_{-\infty}^\infty p(x+L)x^i dx = \int_{-\infty}^\infty p(-x+L)x^i dx = 0, \qquad i \text{ odd},$$

$$\int_{-\infty}^\infty p(x+L)x^i dx = \int_{-\infty}^\infty p(-x+L)x^i dx, \qquad i \text{ even},$$

it follows that $p_1(x)\equiv p(x+L)+p(-x+L)$ is a weight function for $\{\phi_n(x+L)\}$. Assuming that $p(x)$ has been replaced by $p_1(x)$, we find that $p_1(-x)\equiv p_1(x)$, $T(-x)\equiv T(x)$, $S(-x)\equiv S(x)$, and

$$S(x) = e^{K'x^2}\left\{\int_0^x 2K'ze^{-K'z^2}T(z)dz + C\right\}, \qquad K',\ C \text{ constants}.$$

Let

$$T_1(x) = e^{K'x^2}\left\{\int_0^x 2K'ze^{-K'z^2}T(z)dz + C_1\right\},$$

where C_1 is a constant to be determined. Then $T_1(-x)\equiv T_1(x)$, and $\int_{-\infty}^\infty T_1(x)x^i dx=0$, ($i$ odd). If i is even, then integration by parts shows that

$$u(x) \equiv e^{-K'x^2}\int_\infty^x z^i e^{K'z^2}dz = xP_{i-2}(x) + C_2 e^{-K'x^2}\int_\infty^x e^{K'z^2}dz,$$

where $P_{i-2}(x)$ is a polynomial of degree $i-2$ in x, ($i\geqq 2$, C_2 constant). It follows that

$$\int_0^\infty T_1(x)x^i dx = \left[\int_\infty^x z^i e^{K'z^2}dz\left\{\int_0^x 2K'ze^{-K'z^2}T(z)dz + C_1\right\}\right]_0^\infty$$

$$- 2K'\int_0^\infty xT(x)u(x)dx$$

$$= C_2\left\{C_1\int_\infty^x e^{K'z^2}dz\right.$$

$$\left. - 2K'\int_0^\infty xT(x)e^{-K'x^2}\left(\int_\infty^x e^{K'z^2}dz\right)dx\right\}$$

$$= 0, \qquad i \text{ even},$$

if C_1 is properly chosen. Thus $\int_{-\infty}^{\infty} T_1(x)x^i dx = 0$, $(i = 0, 1, \cdots)$. Except for a function whose moments vanish, the weight function reduces to $C_3e^{K'x^2}$, (C_3 an arbitrary constant). Replacing x by $x/(-K')^{1/2}$ and putting $C_3 = 1$, we obtain e^{-x^2}, which is the weight function for Hermite polynomials.

4. **Conclusion.** Having completed a proof of the theorem, we give the following corollary:

COROLLARY. *If* $\{\phi_n(x)\}$ *is an orthogonal system of polynomials which is also an Appell system, so that* $\phi_n'(x) \equiv n\phi_{n-1}(x)$ *(that is,* $p(x) \equiv q(x)$*), then* $\{\phi_n(x)\}$ *is reducible to the system of Hermite polynomials by means of a linear transformation on* x.

Meixner* first proved this result, but other proofs have been given by W. Hahn,† the author,‡ and Shohat.§ Sheffer's|| recurrence relation for Appell polynomials and the recurrence relation for orthogonal polynomials¶ enable us to give a more direct proof.** Comparing Sheffer's relation

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* J. Meixner, *Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion*, Journal of the London Mathematical Society, vol. 9 (1934), pp. 6–13.

† Loc cit.

‡ M. Webster, *On the zeros of Jacobi polynomials with applications*, Duke Mathematical Journal, vol. 3 (1937), pp. 426–442.

§ J. Shohat, *The relation of the classical orthogonal polynomials to the polynomials of Appell*, American Journal of Mathematics, vol. 58 (1936), pp. 453–464.

|| I. Sheffer, *A differential equation for Appell polynomials*, this Bulletin, vol. 41 (1935), pp. 914–923.

¶ J. Shohat, *Théorie Générale des Polynômes Orthogonaux de Tchebichef*, Mémorial des Sciences Mathématiques, vol. 66, Paris, 1934.

** This Bulletin, abstract 42-3-127.

$$\phi_n(x) = (x + b_0)\phi_{n-1}(x) + (n-1)b_1\phi_{n-2} + (n-1)(n-2)b_2\phi_{n-3}(x) + \cdots + (n-1)(n-2)\cdots 1b_{n-1}\phi_0(x)$$

with

$$\phi_n(x) = (x - c_n)\phi_{n-1}(x) - \lambda_n\phi_{n-2}(x),$$

we have

$$c_n = -b_0, \quad b_2 = b_3 = \cdots = b_{n-1} = 0, \quad \lambda_n = -b_1(n-1) > 0,$$

for $n>1$. Let $x=(-2b_1)^{1/2}y-b_0$; then $\phi_n(x)\equiv(-2b_1)^{n/2}\psi_n(y)$, where $\psi_n(y)\equiv y\psi_{n-1}(y)-[(n-1)/2]\psi_{n-2}(y)$, which proves that $\{\psi_n(y)\}$ is the set of Hermite polynomials.

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A CORRECTION

EVERETT PITCHER AND W. E. SEWELL

C. R. Adams and J. A. Clarkson have kindly shown us that in our recent paper* Theorem 2.1 is false. Easy examples show that equation (2.2) may have no solutions or many solutions. In the proposed proof, (2.6) does not follow from (2.5) as stated. The material of the paper can be made correct by strengthening the hypothesis (2.1) and the corresponding hypotheses in the applications. The following changes should be made.

In §1 delete the first sentence of the second paragraph.

In §2 change the statement of Theorem 2.1 so that the first three lines of page 101 read "and such that there is a constant B between 0 and 1 for which, with y_1 and y_2 in E, we have

$$(2.1) \qquad |Sy_1 - Sy_2| \leqq B \max |y_1 - y_2|."$$

This theorem is well known.† The part of §2 following the theorem is to be deleted.

In (4.4), (4.12), (4.14), and (4.15) remove the exponent α from $|y-y'|$ and $|y_1-y_2|$.

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* Everett Pitcher and W. E. Sewell, *Existence theorems for solutions of differential equations of non-integral order*, this Bulletin, vol. 44 (1938), pp. 100–107.

† Compare G. C. Evans, *Functionals and their Applications*, American Mathematical Society Colloquium Publications, vol. 5, New York, 1918, pp. 52–53.