find a suitable mapping of V on the complex plane. This is possible if V can be covered by a finite or denumerable number of neighborhoods homeomorphic to a region in the plane, and if V is orientable. The classification follows the standard procedure, due to Jordan for closed surfaces, and to Kerékjártó for open surfaces.

In Chapter 5, the topological characterization of analytic functions is given. A transformation f of the space X into the space X_0 is called *equivalent* to the transformation f' of X' into X_0' if there are topological transformations h of X into X' and h_0 of X_0 into X_0' such that

$$f'(p) = h_0(f(h^{-1}(p))).$$

Two properties of mappings which are invariant under this equivalence are the following: The image of any open set is an open set, and no closed connected set containing more than one point goes into a single point. Mappings with these properties are called *interior*. (This term, or *inner*, is now commonly used by topologists to refer to mappings satisfying the first property.) The fundamental theorem is that any analytic function, as a mapping of its Riemann surface on the complex plane, is interior, and conversely, any interior mapping of a surface on the complex plane is equivalent to an analytic function for which the given surface can be taken as its Riemann surface. The essential step in the proof is to show that an interior transformation behaves locally like z^n for some n, as in (1).

The last chapter gives some applications of preceding methods and results, especially to properties of transformations of one Riemann surface into another. The formula of Hurwitz, relating the genus and number of boundaries of each surface to the degree of the transformation and the total amount of branching, is given and generalized. Asymptotic and limiting values of an analytic function are discussed.

The book should prove of real interest to anyone wishing to study deeply into the underlying topological properties of Riemann surfaces and analytic functions. On the whole, the exposition is quite clear, though here and there one finds slight errors and omissions of important details.

HASSLER WHITNEY

Opérations Infinitésimales Linéaires. By Vito Volterra and Bohuslav Hostinsky. Paris' Gauthiers-Villars, 1938. 7+238 pp.

The infinitesimal calculus of linear operators was invented by Volterra in 1887, and it is with unusual interest that one opens a volume written fifty years later on this important subject, when one discovers that he is a coauthor.

Consider a linear operator X which is a function $X(t)$ of the time t. If multiplication is taken as the fundamental operation,* then analogy with ordinary functions suggests letting the quotient $X(t+\Delta t)X^{-1}(t)$ measure the "change" in X during the interval from t to $t+\Delta t$, and

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} [X(t + \Delta t)X^{-1}(t)] \tag{1}$$

measure the "rate of change," or "derivative" (more properly, right-derivative) of $X(t)$. It is natural to regard this derivative as a sort of "infinitesimal linear operator," whence the title of the book.

* If addition is taken as the fundamental operation, one gets the (commutative) infinitesimal calculus of vectors, which was discussed by H. Grassmann in 1862, in his *Ausdehnungslehre*, part 2, chaps. 2–4.

Having defined derivatives, one naturally defines an indefinite "product integral" of $X(t)$ (in symbols, $\bar{\int} X(t)dt$) as any function whose derivative exists and is $X(t)$. This definition contains as a special case the usual definition of a solution (or integral) of a system of first-order ordinary linear homogeneous differential equations.

Moreover, the familiar theorem that if $Y(t)$ is one solution then the general solution is $Y(t)C$—obtained by right-multiplying $Y(t)$ by a constant matrix—becomes identified with the theorem that if $y(t)$ is one integral of $x(t)$, then the general integral is $y(t)+C$. Likewise, the proof of the existence of integrals of such differential equations becomes merged with Riemann's constructive proof of the existence of a "definite integral" for all continuous functions.

The above ideas were contained in Volterra's original paper; together with some standard material on the canonical form of (complex) matrices, they suggest the content of Chapters I–VI of *Opérations Infinitésimales Linéaires.*

Chapters VII–XI are devoted to new, but closely related topics: partial differentiation and path integration of functions of several real variables, whose values are linear operators. A necessary and sufficient condition that a path integrand be a total differential is found—this is the usual condition of "complete integrability," and gives a fresh interpretation to the latter. The study of partial derivatives also paves the way for the study of analytic matrix functions $X(z)$ of a complex variable; analogies exist to most parts of the usual theory, including contour integrals, for example.

Chapters XII–XV are consecrated to Volterra's early (1895) applications of this last theory to ordinary complex functions. The monodromie group is shown to be composed of the analogs of residues, and "Fuchs' results on linear differential equations are shown to be the extension of Cauchy's theorem on residues." The many other applications include exhaustive classifications of the singularities of the integrals of linear differential equations of various types.

In the last three chapters (XVI–XVIII), Hostinsky tries to apply Volterra's methods to linear operators on function spaces, and indirectly to the theories of heat and diffusion. These are the most provocative chapters of the book.

In the theory of heat, if $f(x)$ expresses the temperature of a bar of unit length as a function of position at time s, then the temperature at a later time t is a function $g(x)$ obtainable by applying to $f(x)$ a (linear) "integral operator of the first kind,"

$$(2) \qquad G(s, t): \ f(x) \to \int_0^1 G(x, y; s, t) f(y) dy,$$

where $G(x, y; s, t)$ is a Green's function independent of $f(x)$. A similar situation exists in the theories of diffusion and of dependent probabilities (alias stochastic processes).

The linear operator $G(s, t)$ is well known to be the *integral* of Fourier's (linear) differential operator $\partial^2/\partial x^2$, which is conversely the *derivative* of $G(s, t)$. But unhappily, Volterra's definitions do not include this case; for example, $G(s, t)$ has no inverse (Maxwell, *Theory of Heat*, 1872, p. 241), and so (1) is meaningless.

On the other hand, "integral operators of the second kind": $f(x) \to f(x) + \int_0^1 K(x, y) f(y) dy$, can be handled by Volterra's methods. One can define inverses by Neumann's series $(1+X)^{-1} = 1 - X + X^2 - X^3 + \cdots$, and use (1), provided $|K(x, y)| < \frac{1}{2}$. This yields a limited "infinitesimal calculus of infinite matrices," which includes what Khintchine calls the "Poisson case" of dependent probabilities.

Hostinsky's theory of integral operators of the first kind (Chapter XVIII), on the other hand, bears little resemblance to Volterra's theory. It begins with the identity "of Chapman-Kolmogoroff": $G(s, t)G(t, u) = G(s, u)$ if $s < t < u$. Hostinsky points out that this condition expresses the fact that $G(s, t)$ is the definite integral of an "infinitesimal linear operator." Kolmogoroff (Mathematische Annalen, vols. 104, 108) had previously, but in less suggestive language, correlated such families of operators with infinitesimal differential operators; technically, the reviewer prefers Kolmogoroff's treatment.

The reviewer also feels that Hostinsky could have used with profit the modern theory of function spaces (von Neumann, Stone, Banach). Again, von Neumann's theory of groups of linear transformations, and Schlesinger's Lebesgue integration of matrices, might well have been sketched—and a reference to Delsarte's work is really called for.

But these defects do not prevent the book from being of the first importance—in fact, probably the best available introduction to the higher theory of linear differential equations.

GARRETT BIRKHOFF

Économique Rationnelle. By Guillaume and Ed. Guillaume. (Actualités Scientifiques et Industrielles, nos. 504–508.) Paris, Hermann, 1937. 375 pp.

An Econometric Approach to Business Cycle Problems. By J. Tinbergen. (Actualités Scientifiques et Industrielles, no. 525.) Paris, Hermann, 1937. 73 pp.

The great increase of the last few years in statistical material relating to the phenomena of economics has brought realization that the "laws of economics" must be subjected to new scrutiny. In some cases they must be modified, in others reformulated with the introduction of new variables. In this modern adventure, to which the name of econometrics has been generically given, mathematics plays its customary central role. Although the origin of mathematical economics antedates the publication in 1838 of A. A. Cournot's classic *Theory of Riches*, which ushered in the justly celebrated works of Jevons, Walras, Marshall, Edgeworth, Pareto, and others of the mathematical school, econometrics in the modern sense is a new science. The books before us for review indicate the trend that modern studies are taking.

The first volume is divided into five parts: (1) Method, (2) Pure economy, (3) Pure economy in interference with the legal domain, (4) The national legal-economic domain in interference with external society, (5) Mathematical models of the economic world.

It is the last part which will be of the most interest to mathematicians, since the authors attempt to establish mathematical models for the interpretation of economic phenomena. Their point of view is taken from dynamics. The authors set up equations which represent for them "the principle of the conservation of the flux of commodities." They introduce the notion of "money liquidity" in a theory of "kinetic and potential money." Economic equilibrium conditions are formulated in terms of the differential variation in commodities. Intriguing new units are employed such as the gold-gram as a quantity of value, the gold-gram/sec. as the flux of value, the gold-gram/man-sec. as the density of flux.

The fundamental tenet of the authors is found in the statement: "We shall see that in attempting to create a *rational economics* similar to a body of doctrine such as rational mechanics, we are led similarly to search for some principle of conservation, more particularly the axiom which we shall call the 'principle of the conservation of value.' "

The prefatory material is elegantly expressed and contains much of interest to the philosophy of science. Bridgman's operational theory is regarded as an essential viewpoint for the new methods. Thus, in criticising the subjective theories of the older economists, the authors remark: "Unfortunately, their efforts have remained