ALGEBRAIC POSTULATES AND A GEOMETRIC INTER-PRETATION FOR THE LEWIS CALCULUS OF STRICT IMPLICATION

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1. Two further postulates for a Boolean ring with a unit element. If addition, subtraction, and multiplication are properly defined in logic, it may be shown* that the postulates for these operations are identical with those in a ring, in which every element is idempotent, satisfying the postulate $xx = x$. Such a ring is called a Boolean ring. The postulates for a Boolean ring with a unit element are therefore the following:

A. *Addition is always possible, commutative, and associative.*

B. *Multiplication is always possible, associative, and both left- and right-distributive with respect to addition.*

C. *Subtraction is always possible.*

D. $xx = x$.

E. *There exists an element* 1 *such that* $x1 = x$ *for every element* x *in the ring.*

Here we shall introduce a new operation, represented by x^∞, which satisfies the following two further postulates:

F_1. *For every element x there exists an element x^∞ such that $x^\infty x = x^\infty$.*

F_2. *For any two elements x and y we have $(xy)^\infty = x^\infty y^\infty$.*

The postulates A–F_2, obtained above, may be called the algebraic postulates for the Lewis calculus of strict implication.

2. A geometric meaning of the symbol x^∞. A geometric meaning†
may be attached to x^∞ as follows: Let x be a point set in the euclidean

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* See M. H. Stone. *The theory of representations for Boolean algebras*, Transactions of this Society, vol. 40 (1936), pp. 37–53.

† Another geometric meaning of x^∞ may be obtained by assuming 1^∞ to be any one fixed point or any set of fixed points (finite or infinite in number and continuous or discontinuous in character) and setting $x^\infty = x1^\infty$. If we assume that 1^∞ is a fixed point, we have then the following property:

G. x^∞ *is two-valued, that is, $x^\infty = 1^\infty$ or 0^∞,*

which is independent of the postulates A–F_2. This sub-Boolean algebra with the postulates A–G does not become the ordinary two-valued Boolean algebra, unless we assume further that x is two-valued.

plane; then x^∞ may be taken as the interior points of x (the set of all points p such that some circle with center at p lies entirely within x). Then the postulates F_1 and F_2 are satisfied. Moreover $(1-x)^\infty$ is the set of points interior to the complement of x, that is, exterior of x; $1-x^\infty-(1-x)^\infty$ is the frontier of x; and $1-(1-x)^\infty$ is the closure of x, that is, x plus its frontier.

3. **A geometric interpretation of Lewis' possibility functions.** When p is a proposition, Lewis introduces an undefined idea $\Diamond p$, which is read "p is possible" and may be called the possibility function of p. Now, if a class x be given, $\Diamond x$ must have a corresponding meaning which is obtained from the following definition:

DEFINITION. $\Diamond x = 1-(1-x)^\infty$.

By means of the definition of $\sim x$, we obtain $\sim x = 1-x$; and we have the following theorems:

THEOREM 1. $\Diamond x$ is the closure of x.

THEOREM 2. $\sim\Diamond x = (1-x)^\infty$ is the exterior of x.

THEOREM 3. $\Diamond\sim x = 1-x^\infty$ is the closure of the complement of x.

THEOREM 4. $\sim\Diamond\sim x = x^\infty$ is the interior of x.

On the basis of the postulates A-F we may then prove abstractly the following theorems:

THEOREM 5. $0^\infty = 0$ and $\sim\Diamond\sim 0. = .0$; that is, the interior of the null class is a null class.

THEOREM 6. $\sim\Diamond\sim x.\sim\Diamond x: = .0$, and $x^\infty(\sim x)^\infty = 0$; that is, if x is a class, then the interior and the exterior of x have no point in common.

DEFINITION. $x^f = 1-x^\infty-(\sim x)^\infty$, that is, x^f is the frontier of x.

THEOREM 7. $x^f = (\sim x)^f$, that is, x and $\sim x$ have the same frontier.

THEOREM 8. $x^f + x^\infty + (\sim x)^\infty = 1$, $x^f + (\sim\Diamond\sim x) + (\sim\Diamond x) = 1$; that is, the frontier, the interior, and the exterior of x form the whole plane.

THEOREM 9. $\Diamond x. = .x^f + x^\infty$ and $\Diamond x. = .x^f + (\sim\Diamond\sim x)$; that is, $\Diamond x$ is the sum of the frontier and the interior of x.

THEOREM 10. $\Diamond\sim x. = .x^f + (\sim x)^\infty$ and $\Diamond\sim x. = .x^f + \sim\Diamond x$; that is, $\Diamond\sim x$ is the sum of the frontier and the exterior of x.

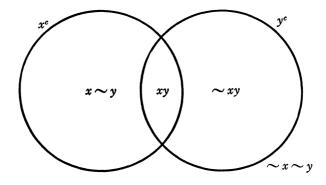THEOREM 11. $x(\sim x)^\infty = 0$ and $x.\sim\Diamond x: = .0$; that is, the class x and its exterior have no point in common.

THEOREM 12. $x = x^\infty + xx^f$ and $x = (\sim\diamond\sim x) + xx^f$; that is, the class x is the sum of its interior and the part of its frontier which belongs to it.

THEOREM 13. $(\sim x) = (\sim x)^\infty + (\sim x)x^f$ and $(\sim x) = (\sim\diamond x) + (\sim x)x^f$; that is, the class $\sim x$ is the sum of the exterior of x and the part of its frontier which belongs to $\sim x$.

4. **Theorems corresponding to Lewis' postulates.** In *Symbolic Logic* by Lewis and Langford the symbol $p\dashv q$ is defined by $p\dashv q. = .\sim\diamond(p\sim q)$. Changing the propositions p and q into the classes x and y, we get the definition of $x\dashv y$:

DEFINITION. $x\dashv y. = .\sim\diamond(x\sim y)$.

The geometric meaning of $x\dashv y$ is very clear, for its definition states that $x\dashv y$ means the exterior of $x\sim y$; that is, if, for example, two curves x^c and y^c divide the plane into four parts xy, $\sim xy$, $x\sim y$, and $\sim x\sim y$, as in the figure, then $x\dashv y$ means the interior of $x\supset y$, which is the sum of xy, $\sim xy$, and $\sim x\sim y$. In other words, we have the following two theorems:

THEOREM 14. $x\supset y. = .xy + (\sim xy) + (\sim x\sim y)$.*

THEOREM 15. $x\dashv y. = .\sim\diamond\sim(x\supset y). = .(x\supset y)^\infty$.

PROOF. By the definition of $\sim x$, $\sim x. = .1 - x$, and we see that

$$xy + (\sim xy) + (\sim x\sim y) = xy + (1-x)y + (1-x)(1-y)$$
$$= xy + y - xy + 1 - x - y + xy$$
$$= 1 - x + xy$$
$$= (x\supset y).$$

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* $x\supset y$ is, of course, defined as $\sim(x\sim y)$ which is $1-x+xy$.

By Theorem 4

$$\sim \diamond \sim (x \supset y) . = . (x \supset y)^{\infty},$$

$$\sim \diamond \sim (x \supset y) . = . \sim \diamond \sim (1 - x + xy)$$

$$= . \sim \diamond \left[1 - (1 - x + xy) \right]$$

$$= . \sim \diamond (x - xy)$$

$$= . \sim \diamond x(1 - y)$$

$$= . \sim \diamond (x \sim y)$$

$$= . x \dashv 3 y.$$

DEFINITION. $i = (\sim \diamond 0)$.

This definition of i may be replaced by the following theorem:

THEOREM 16. $i = 1^{\infty}$ and $i = (\sim\diamond\sim1)$.

PROOF. We have $i = (\sim\diamond 0) = [\sim\diamond(1-1)] = (\sim\diamond\sim1) = 1^{\infty}$, by Theorem 3.

THEOREM 17. $x \dashv 3 x . = . i$.

PROOF. We have $x \dashv 3 x . = . \sim\diamond(x\sim x) = . \sim\diamond\ 0 = . i$.

If p and q are propositions, then we have the following theorem:[*]

If $p \dashv 3 q$ is asserted, then $p \dashv 3 q . = . i$.

Lewis' postulates may therefore be written as follows:

$$pq \dashv 3 qp . = . i,$$

$$pq \dashv 3 p . = . i,$$

$$p \dashv 3 pp . = . i,$$

$$(pq)r \dashv 3 p(qr) . = . i,$$

$$p \dashv 3 \sim (\sim p) . = . i,$$

$$p \dashv 3 q . q \dashv 3 r : \dashv 3 . p \dashv 3 r : = . i,$$

$$p . p \dashv 3 q : \dashv 3 . q : . = . i,$$

$$\diamond (pq) \dashv 3 \diamond p . = . i.$$

The corresponding theorems for classes are obtained by changing the propositions p, q, r into x, y, z, respectively:

THEOREM 18. $xy \dashv 3 yx . = . i$.

PROOF. We have $xy \dashv 3 yx . = . xy \dashv 3 xy = . i$ by Theorem 17.

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[*] See this Bulletin, vol. 42 (1936), p. 708, Theorem 9.

THEOREM 19. $xy \dashv_3 x. = .i.$

PROOF. We have $xy \dashv_3 x. = . \sim \Diamond (xy \sim x) = . \sim \Diamond (y0) = . \sim \Diamond \ 0 = .i.$

In like manner we may prove Theorems 20–22 below:

THEOREM 20. $x \dashv_3 xx. = .i.$

THEOREM 21. $(xy)z \dashv_3 x(yz). = .i.$

THEOREM 22. $x \dashv_3 \sim(\sim x). = .i.$

THEOREM 23. $x \dashv_3 y. y \dashv_3 z : \dashv_3 .x \dashv_3 z : . = .i.$

PROOF. We have

$$x \dashv_3 y. y \dashv_3 z : \dashv_3 . x \dashv_3 z : . = : (x \supset y)^\infty (y \supset z)^\infty . \dashv_3 . x \dashv_3 z$$
$$= : (x \supset y. y \supset z)^\infty . \dashv_3 . x \dashv_3 z$$
$$= : (x \supset y. y \supset z. x \supset z)^\infty . \dashv_3 . x \dashv_3 z$$
$$= : (x \supset y)^\infty (y \supset z)^\infty (x \supset z)^\infty . \dashv_3 . x \dashv_3 z$$
$$= : . x \dashv_3 y. y \dashv_3 z. x \dashv_3 z : \dashv_3 . x \dashv_3 z$$
$$= : . x \dashv_3 z. x \dashv_3 y. y \dashv_3 z . : \dashv_3 . x \dashv_3 z$$
$$= . i,$$

where the first five equations can be obtained, respectively, from Theorem 15, F_2, Theorem 23.1 (below), F_2, and Theorems 4 and 15. The last equation follows from Theorem 19.

THEOREM 23.1. $x \supset y. y \supset z : = : x \supset y. y \supset z. x \supset z.$

PROOF. We have

$$x \supset y. y \supset z : = : (\sim x + xy). (\sim y + yz)$$
$$= : \sim x \sim y + \sim xyz + xy \sim y + xyyz$$
(1)
$$= : \sim x \sim y + \sim xyz + 0 + xyz$$
$$= : \sim x \sim y + yz.$$

Then

$$x \supset y. y \supset z. x \supset z. : = : (\sim x \sim y + yz). (x \supset z)$$
$$= : (\sim x \sim y + yz). (\sim x + xz)$$
$$= : \sim x \sim x \sim y + \sim xyz + 0 + xyzz$$
$$= : \sim x \sim y + \sim xyz + xyz$$
$$= : \sim x \sim y + yz$$
$$= : x \supset y. y \supset z,$$

by (1) above.

THEOREM 24. $x . x \dashv 3 \, y : \dashv 3 \, . y : . = . i.$

PROOF. We have

$$x . x \dashv 3 \, y : \dashv 3 . y : . = . \sim \Diamond \, (x . x \dashv 3 \, y . \sim y)$$
$$= . \sim \Diamond \, (x \sim y . x \dashv 3 \, y)$$
$$= . \sim \Diamond \, (x \sim y . \sim \Diamond \, (x \sim y))$$
$$= . \sim \Diamond \, 0$$
$$= . i,$$

by Theorem 11 and a definition.

THEOREM 25. $\Diamond(xy) \dashv 3 \, \Diamond x . = . i.$

PROOF. By the application of Theorem 25.1 (below),

$$\Diamond(xy) \dashv 3 \, \Diamond \, x . = . \sim \Diamond \, [\Diamond \, (xy) \sim \Diamond \, x]$$
$$= . \sim \Diamond \, [\Diamond(xy) \sim \Diamond \, x \sim \Diamond \, (xy)]$$
$$= . \sim \Diamond \, (\sim \Diamond \, x . 0)$$
$$= . \sim \Diamond \, 0$$
$$= . i,$$

by definition.

THEOREM 25.1. $\sim\Diamond x . = : \sim\Diamond x . \sim\Diamond(xy).$

PROOF. We have, using F_2,

$$\sim \Diamond \, x . \sim \Diamond \, (xy) : = . (1 - x)^{\infty} (1 - xy)^{\infty},$$
$$= . [(1 - x)(1 - xy)]^{\infty}$$
$$= . (1 - x - xy + xxy)^{\infty}$$
$$= . (1 - x)^{\infty}$$
$$= . \sim \Diamond \, x,$$

where the first step follows by Theorem 1 and the last by Theorem 2.

5. **Theorems corresponding to Lewis' operations.** Lewis' substitution can be performed both on propositions and on classes. We consider therefore only adjunction and inference. For this purpose, we first establish the following two theorems:

THEOREM 26. $x . x \dashv 3 \, y : = : xy . x \dashv 3 \, y.$

PROOF. By Theorem 11 we have

$$x \sim y . \sim \Diamond \, (x \sim y) : = .0,$$

or

$$x(1 - y) . x \dashv 3 \; y : = .0,$$

or

$$(x - xy) . x \dashv 3 \; y : = .0,$$

or

$$(x . x \dashv 3 \; y) - (xy . x \dashv 3 \; y) = 0,$$

from which we get at once Theorem 26.

THEOREM 27. (1) *If* $i \dashv 3 \; x . = .i$, *then* $ix = i$. (2) *Conversely, if* $ix = i$, *then* $i \dashv 3 \; x . = .i$.

PROOF. We obtain, using Theorem 16, F_2, and Theorem 26, $ix = 1^\infty x = (1.1)^\infty x = 1^\infty 1^\infty x = iix = ixi = ix(i \dashv 3 \; x) = i(i \dashv 3 \; x) = ii = i$.

The proof of the converse theorem is the following: We have $i \dashv 3 \; x . = . \sim \Diamond (i \sim x) = . \sim \Diamond \, [i \sim (ix)] = . \sim \Diamond (i \sim i) = . \sim \Diamond \; 0 = .i$, by definition.

DEFINITION. *We write* $\vdash x$, *when and only when* $ix = i$.

Thus, we may write $\vdash x$, when $x = 1$ or $x = i$. For example, Theorem 18 is equivalent to $\vdash . xy \dashv 3 \; yx$.

We can now prove the theorems corresponding to adjunction and inference.

THEOREM 28. *If* $\vdash x$ *and* $\vdash y$, *then* $\vdash (xy)$.

PROOF. We have $i(xy) = iixy = ixiy = ii = i$. Hence by definition, $\vdash (xy)$.

THEOREM 29. *If* $\vdash x$, *and* $\vdash . x \dashv 3 \; y$, *then* $\vdash y$.

PROOF. We obtain $iy = iiy = (ix)(i . x \dashv 3 \; y)y = ixi(x \dashv 3 \; y)y = iixy(x \dashv 3 \; y)$ $= iix(x \dashv 3 \; y) = (ix)i(x \dashv 3 \; y) = ii = i$, using Theorem 26. Therefore $iy = i$; hence by definition $\vdash y$.

6. **Miscellaneous theorems.** We state the following theorems, each without proof:

THEOREM 30. $x \dashv 3 \; y . = : i . x \dashv 3 \; y$.

THEOREM 31. *If* $x \dashv 3 \; y . = .i$, *and* $y \dashv 3 \; z . = .i$, *then* $x \dashv 3 \; z . = .i$.

THEOREM 32. *If* $x \prec_3 y. = .i$ *and* $x \prec_3 y . \prec_3 . u \prec_3 v := .i$, *then* $u \prec_3 v. = .i$.

THEOREM 33. (1) *If* $x \prec_3 y. = .i$, *then* $\vdash .x \prec_3 y$. (2) *Conversely, if* $\vdash .x \prec_3 y$, *then* $x \prec_3 y. = .i$.

THEOREM 34. (1) *If* $i \prec_3 x. = .i$, *then* $\vdash x$. (2) *Conversely, if* $\vdash x$, *then* $i \prec_3 x. = .i$.

THEOREM 35. *If* $1 \prec_3 x. = .i$, *then* $\vdash x$.

7. A relation between Huntington's "p in T" and "$\vdash p$" in this paper. In this Bulletin (vol. 40 (1934), p. 733) E. V. Huntington established the theorem

(2) $$(p < q) \rightleftarrows [(p \prec_3 q) \ in \ T]$$

from the following definition of $(p < q)$:

(3) $$(p < q) \rightleftarrows (p = pq).$$

It is now my aim to prove the following theorem:

THEOREM 36. $[(p \prec_3 q) \ in \ T] \rightleftarrows (\vdash .p \prec_3 q)$.

PROOF OF $[(p \prec_3 q) \ in \ T] \rightarrow (\vdash .p \prec_3 q)$. By hypothesis, $(p \prec_3 q)$ in T. By (2) and (3), $p = pq$. Therefore, $p \prec_3 q. = .pq \prec_3 q. = .i$. Thus $\vdash .p \prec_3 q$.

PROOF OF $(\vdash .p \prec_3 q) \rightarrow [(p \prec_3 q) \ in \ T]$. By (2) and (3) $(p = pp) \rightarrow [(p \prec_3 p) \ in \ T]$. Therefore, $(p \prec_3 p)$ in T, that is,

(4) $$i \ in \ T.$$

But by hypothesis, $i . p \prec_3 q := .i$. Hence

(5) $$p \prec_3 q. = .i.$$

From (4) and (5) we have at once $(p \prec_3 q)$ in T.

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