POLYGENIC FUNCTIONS WHOSE ASSOCIATED ELEMENT-TO-POINT TRANSFORMATION CONVERTS UNIONS INTO POINTS*

EDWARD KASNER

1. **Introduction.** A function $w=\phi(x, y)+i\psi(x, y)$ is called a *polygenic function* of the complex variable $z=x+iy$ if the real functions ϕ and ψ are general, that is, are not required to satisfy the Cauchy-Riemann equations. The value of the derivative of a polygenic function at a point z_0 depends in general not only on the point z_0 but also on the direction θ along which z approaches z_0; that is, dw/dz is of the form $F(x, y, \theta)$. Thus the derivative $\gamma=dw/dz$ of a polygenic function may be regarded as determining a correspondence between the lineal elements (x, y, θ) of the z-plane and the points (α, β) of the γ-plane, where $\gamma=\alpha+i\beta$. We call this correspondence the *element-to-point transformation T associated with the polygenic function w.*

In previous papers (Kasner, *A new theory of polygenic functions*, Science, vol. 66 (1927), pp. 581–582; *General theory of polygenic functions*, Proceedings of the National Academy of Sciences, vol. 13 (1928), pp. 75–82; *The second derivative of a polygenic function*, Transactions of this Society, vol. 30 (1928), pp. 805–818) we have shown that the element-to-point transformation T associated with a polygenic function must possess the two following properties:

I. *Elements at a given point in the z-plane correspond to points of a circle I in the γ-plane.*

II. *Corresponding central angles of the circle and angles at the point are in the ratio* $-2:1$.

If an element-to-point transformation T possesses the property I, then we define the function $H+iK$, which as a vector represents the center of the circle I, to be the *center function* of T, and the function $(H+h)+i(K+k)$, which as a vector represents the point (called the *initial point* of the circle) on the circle I which corresponds to the initial direction $\theta=0$ in the z-plane, we define to be the *principal phase function* of T. The circle I together with its initial point we call a *clock*.

We then find (Kasner, *A complete characterization of polygenic functions*, Proceedings of the National Academy of Sciences, vol. 22

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(1936), pp. 172–177), that the element-to-point transformation T associated with a polygenic function possesses the following additional property:

III. *The principal phase point of the clock representing the derivative of the center function of T coincides with the center of the clock representing the derivative of the principal phase function of T.*

In the paper last cited, it is also proved that *for an element-to-point transformation to be associated with a polygenic function, it is necessary and sufficient that it possess the properties* I, II, *and* III.

The associated transformation T carries a single element into a point, and it carries the ∞^1 elements at a point in the z-plane into the points of a circle in the γ-plane (property I). However, a given point in the γ-plane will correspond, in general, not to a single element in the z-plane, but to a series (∞^1 elements). Now we inquire *under what conditions will this series be a union (curve or point)*. Of course we mean that this shall happen for all the points of the γ-plane, that is, we demand that *all* the series so formed shall be unions. It turns out analytically that this problem means that a certain pair of functions of (x, y, p) shall be *in involution*. In our discussion, we do not demand that the jacobians be different from zero; therefore our solution will include degenerate cases. But actually the major part of the solution is not degenerate.

Our problem is thus to determine a certain specific class of polygenic functions, namely, that class for which, instead of associated series, we obtain unions. *This class, we find by a long analytic discussion, consists of the following three distinct types*:

(A) *The monogenic functions* $w=f(z)$.

(B) *The mixed quadratic fractional polygenic functions*

$$w = -\frac{az+b}{\bar{a}(\bar{a}\bar{z}+\bar{b})} + cz + d, \qquad a \neq 0.$$

(C) *The affine linear polygenic functions* $w=Az+B\bar{z}+C$, $(B\neq 0)$.

Of these three types, the quadratic type (B) is the essentially significant result revealed by our investigation.

2. **The associated element-to-point transformation T of a polygenic function.** Let the element-to-point transformation T

$$\gamma = \alpha(x, y, \theta) + i\beta(x, y, \theta) \tag{1}$$

possess the properties I, II, and III. Then we find that T can be written in the form

$$\alpha = H + h \cos 2\theta + k \sin 2\theta, \qquad (2)$$
$$\beta = K - h \sin 2\theta + k \cos 2\theta,$$

where H, K, h, k are functions of x and y only which satisfy

$$H_x - K_y = h_x + k_y, \quad K_x + H_y = k_x - h_y. \qquad (3)$$

Let $w = \phi(x, y) + i\psi(x, y)$ be any polygenic function to which T is the associated element-to-point transformation. Then w must satisfy the two equations

$$\frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] w = H + iK, \quad \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] w = h + ik. \qquad 4)$$

From (4) it can very easily be proved that *any two polygenic functions which have the same associated element-to-point transformation T differ merely by a complex constant.*

3. **Polygenic functions whose associated element-to-point transformations convert unions into points.** We prove the following theorem:

THEOREM. *The totality of polygenic functions whose associated element-to-point transformations convert unions of the z-plane into the points of the γ-plane consists of the three distinct types* (A), (B), (C), *as specified at the end of* §1.

The proof will occupy the next three pages. Upon writing $p = \tan \theta$ the equations (2) become

$$\alpha = \frac{(H + h) + 2kp + (H - h)p^2}{1 + p^2}, \qquad (5)$$
$$\beta = \frac{(K + k) - 2hp + (K - k)p^2}{1 + p^2}.$$

First let us consider the case in which h and k are both zero. Then from (5) we see that our element-to-point transformation becomes a point-to-point transformation. Hence when $h = k = 0$, the points of the z-plane become the points of the γ-plane, and the condition of onr theorem is therefore satisfied. From (4) we find that w must be a monogenic function of z. Henceforth we shall suppose that at least one of the functions h and k is different from zero.

For the element-to-point transformation (5) to convert unions of the z-plane into the points of the γ-plane it is necessary and sufficient that

$$\frac{\beta_x + p\beta_y}{\alpha_x + p\alpha_y} = \frac{\beta_p}{\alpha_p}; \tag{6}$$

that is, the functions α and β must be in involution. Substituting (5) into (6) and making use of the equations (3), we obtain

$$\frac{(K_x + k_x) + p(H_x - 3h_x) + p^2(-H_y - 3h_y) + p^3(K_y - k_y)}{(H_x + h_x) + p(-K_x + 3k_x) + p^2(K_y + 3k_y) + p^3(H_y - h_y)} = \frac{h + 2kp - hp^2}{-k + 2hp + kp^2}. \tag{7}$$

Since the equation (7) is an identity in p, we obtain, upon setting the coefficients of the powers of p equal to zero and making use of the equations (3), the equations

$$\begin{aligned} \frac{h}{k} &= \frac{-K_x - k_x}{H_x + h_x} = \frac{3H_x - h_x}{3K_x - k_x} = \frac{H_y + k_x}{K_y - h_x} \\ &= \frac{H_x + k_y}{K_x - h_y} = \frac{3H_y + h_y}{3K_y + k_y} = \frac{K_y - k_y}{-H_y + h_y}. \end{aligned} \tag{8}$$

From the equations (8) it follows by ratio and proportion that

$$\frac{h}{k} = \frac{H_y - K_x}{H_x + K_y} = \frac{H_x + K_y}{-H_y + K_x}. \tag{9}$$

Since all the functions are real, it follows from (9) that

$$H_x = -K_y, \qquad H_y = K_x. \tag{10}$$

From (10) we find that $H+iK$ is an analytic function of $\bar{z}$; that is

$$H + iK = \lambda(\bar{z}), \tag{11}$$

where $\lambda(\bar{z})$ is an analytic function of $\bar{z}$.

Substituting $K_x = H_y$ and $K_y = -H_x$ into (8), we find that the equations (8) become

$$\frac{h}{k} = \frac{-H_y - k_x}{H_x + h_x} = \frac{3H_x - h_x}{3H_y - k_x} = \frac{H_x + k_y}{H_y - h_y} = \frac{3H_y + h_y}{-3H_x + k_y}. \tag{12}$$

Also substituting $K_x = H_y$ and $K_y = -H_x$ into equations (3), we obtain

$$2H_x = h_x + k_y, \qquad 2H_y = -h_y + k_x. \tag{13}$$

Upon substituting (13) into (12), we find

$$\frac{h}{k} = \frac{h_y - 3k_x}{3h_x + k_y} = \frac{h_x + 3k_y}{-3h_y + k_x}. \tag{14}$$

These two equations are equivalent to the equations

$$\begin{aligned} 3hh_x + 3kk_x &= kh_y - hk_y, \\ 3hh_y + 3kk_y &= hk_x - kh_x. \end{aligned} \tag{15}$$

The equations (15) are then equivalent to the equations

$$\begin{aligned} \frac{3}{2}\frac{\partial}{\partial x}\log(h^2 + k^2) &= -\frac{\partial}{\partial y}\arc\tan k/h, \\ \frac{3}{2}\frac{\partial}{\partial y}\log(h^2 + k^2) &= \frac{\partial}{\partial x}\arc\tan k/h. \end{aligned} \tag{16}$$

From (16), it follows that $(3/2)\log(h^2+k^2)+i\ \mathrm{arc}\tan k/h$ is an analytic function of $\bar{z}$. Thence $\exp\{(3/2)\log(h^2+k^2)+i\ \mathrm{arc}\tan k/h\}$ is an analytic function of $\bar{z}$; that is

$$(h^2 + k^2)(h + ik) = \mu(\bar{z}), \tag{17}$$

where $\mu(\bar{z})$ is an analytic function of $\bar{z}$. Moreover $\mu(\bar{z})\neq 0$, since at least one of the quantities h, k is different from zero.

Now (17) may be written in the form

$$(h - ik)(h + ik)^2 = \mu(\bar{z}). \tag{18}$$

Upon taking the conjugate of the equation (18), we obtain

$$(h + ik)(h - ik)^2 = \bar{\mu}(z). \tag{19}$$

Solving the equations (18) and (19) for $h+ik$, we find

$$h + ik = [\mu(\bar{z})]^{2/3}/[\bar{\mu}(z)]^{1/3}. \tag{20}$$

It is seen that the condition of our theorem is satisfied if the four functions H, K, h, k satisfy the equations (3), (11), and (20). From these equations, we find that w must be an analytic polygenic function of x and y. Hence w may be written as an analytic function of z and $\bar{z}$; that is

$$w = f(z, \bar{z}), \tag{21}$$

where f is an analytic function of z and $\bar{z}$.

From equations (4), (11), (20), and (21), we find that

$$f_z = \lambda(\bar{z}), \qquad f_{\bar{z}} = [\mu(\bar{z})]^{2/3}/[\bar{\mu}(z)]^{1/3}. \tag{22}$$

The equations (3) are then equivalent to

$$\frac{\lambda'(\bar{z})}{[\mu(\bar{z})]^{2/3}} = \frac{d}{dz}\,[\bar{\mu}(z)]^{-1/3}. \tag{23}$$

From (23) we find that

$$\frac{\lambda'(\bar{z})}{[\mu(\bar{z})]^{2/3}} = a, \qquad \frac{d}{dz}\,[\bar{\mu}(z)]^{-1/3} = a, \tag{24}$$

where a is a complex constant. From (24) we obtain

$$\lambda'(\bar{z}) = \frac{a}{(\bar{a}\bar{z} + \bar{b})^2}, \qquad \mu(\bar{z}) = \frac{1}{(\bar{a}\bar{z} + \bar{b})^3}\cdot \tag{25}$$

First let us suppose that a is zero. Then from (22) and (25) we find that f_z and $f_{\bar{z}}$ are both constants. Thus w is the affine linear polygenic function $w = Az + B\bar{z} + C$ where $B \neq 0$.

Next let $a \neq 0$. Then from (22) and (25) we find that

$$f_z = -\frac{a}{\bar{a}(\bar{a}\bar{z} + \bar{b})} + c, \qquad f_{\bar{z}} = \frac{az + b}{(\bar{a}\bar{z} + \bar{b})^2}\cdot \tag{26}$$

From (26) we see that our polygenic function w must be the mixed quadratic fractional polygenic function

$$w = -\frac{az + b}{\bar{a}(\bar{a}\bar{z} + \bar{b})} + cz + d, \tag{27}$$

where $a \neq 0$, b, c, d are complex constants. This completes the proof.

4. **The unions which under the associated element-to-point transformation become points.** We consider the three classes of functions mentioned in the theorem.

(A) *The monogenic functions* $w = f(z)$. Let the monogenic function $w = f(z)$ be not an affine linear monogenic function. Then the elements at any point z of the z-plane are converted into a point γ of the γ-plane and conversely. *Thus, for a monogenic function which is not affine linear, the* ∞^2 *point-unions of the z-plane are converted into the* ∞^2 *points of the* γ*-plane, and conversely.*

On the other hand, let w be an affine linear monogenic function. Then the derivative of w is constant; hence in the γ-plane we have a single fixed point. To this fixed point corresponds the *opulence* (the totality of ∞^3 elements) of the z-plane. *Thus for an affine linear monogenic function, the opulence of the z-plane is converted into a fixed point of the* γ*-plane.*

In the geometry of lineal elements of the plane, a set of ∞^1 elements is called a *series*, a set of ∞^2 elements is called a *field*, and the totality of ∞^3 elements is called the *opulence*.

(B) *The mixed quadratic fractional polygenic functions*

$$w = -\frac{az+b}{\bar{a}(\bar{a}\bar{z}+\bar{b})} + cz + d.$$

The unions in the z-plane, which under the associated element-to-point transformation of the polygenic function w become the points of the γ-plane, are *the ∞^2 circles through the point $-b/a$ and the field defined by the ∞^1 straight lines through the point $-b/a$.*

To a point $\gamma \neq c$ of the γ-plane there corresponds a definite circle of the z-plane through the point $-b/a$, and conversely. The center C and the radius R are given by the formulas

$$C = -\frac{b}{a} + \frac{\bar{a}}{a^2(\bar{c}-\bar{\gamma})}, \qquad R^2 = \frac{1}{a\bar{a}(c-\gamma)(\bar{c}-\bar{\gamma})}.$$

The field defined by the pencil of straight lines through the point $-b/a$ of the z-plane is converted into the point c of the γ-plane.

(C). *The affine linear polygenic functions* $w = Az + B\bar{z} + C$, $(B \neq 0)$. The associated element-to-point transformation of the affine linear polygenic function $w = Az + B\bar{z} + C$, $(B \neq 0)$, converts the *opulence* (the totality of ∞^3 elements) of the z-plane into the ∞^1 points of the circle in the γ-plane whose center is A and whose radius is $|B| \neq 0$.

It is found that *to any point of the fixed circle in the γ-plane, there corresponds the field defined by ∞^1 parallel straight lines, and conversely.*

5. **Scholium.** We thus find that there are four distinct geometric possibilities in the z-plane:

(A′). The ∞^2 point-unions (stars).

(A″). The *opulence* of elements in the z-plane.

(B). The ∞^2 circles through a fixed point together with the field defined by the pencil of straight lines through the same fixed point.

(C). The ∞^1 fields defined by parallel straight lines.

In the γ-plane, we find the following three distinct geometric possibilities:

(A′, B). The ∞^2 points.

(C). The ∞^1 points of a fixed circle.

(A″) A single fixed point.

We remark in conclusion that the quadratic type (B), formula (27), gives the really significant configuration.

COLUMBIA UNIVERSITY