# A RELATIVE OF THE LEMMA OF SCHWARZ*

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1. **Introduction.** Let $w=f(z)$, where $z=u+iv$, be analytic for $|z|<1$, and let

$$d(r,\theta;f') = |f(re^{i\theta}) - f(0)| = \left|\int_0^r f'(\rho e^{i\theta})d\rho\right|,$$

so that $d(r,\theta;f')$ is the length of the segment on the $w$-plane between the image of the point $z=0$ and the image of the point $z=re^{i\theta}$.

The lemma of Schwarz is the following:

THEOREM 1. *Let $w=f(z)$ be analytic for $|z|<1$. If*

$$d(r,\theta;f') \leqq 1$$

*for all $(r,\theta)$ with $r<1$, then*

$$d(r,\theta;f') \leqq r \tag{1}$$

*and*

$$|f'(0)| \leqq 1. \tag{2}$$

*The sign of equality holds in* (1) (*for $r\neq 0$*) *and in* (2), *if and only if $|f'(z)|\equiv 1$; that is, if and only if the transformation $w=f(z)$ is a rigid motion.*

If the (real) function $g(z)$ is subharmonic for $|z|<1$, then the Lebesgue integral

$$l(r,\theta;g) = \int_0^r g(\rho e^{i\theta})d\rho$$

actually exists. We shall prove the following theorem:

THEOREM 2. *Let $g(z)$ be subharmonic for $|z|<1$. If*

$$l(r,\theta;g) \leqq 1 \tag{3}$$

*for all $(r,\theta)$ with $r<1$, then*

$$l(r,\theta;g) \leqq r \tag{4}$$

*and*

$$g(0) \leqq 1. \tag{5}$$

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*The sign of equality holds in* (4) (*for* $r \neq 0$) *and in* (5), *if and only if* $g(z) \equiv 1$.

In particular, for the function $g(z) \equiv |f'(z)|$, where $f(z)$ is analytic for $|z| < 1$,

$$l(r, \theta; |f'|) = \int_0^r |f'(\rho e^{i\theta})| d\rho$$

is the length of the image on the $w$-plane of the segment between the points $z = 0$ and $z = re^{i\theta}$. The following theorem, which is a special case of Theorem 2, appears to be rather closely related to the lemma of Schwarz:

THEOREM 3. *Let* $w = f(z)$ *be analytic for* $|z| < 1$. *If*

$$l(r, \theta; |f'|) \leqq 1$$

*for all* $(r, \theta)$ *with* $r < 1$, *then*

$$l(r, \theta; |f'|) \leqq r \tag{6}$$

*and*

$$|f'(0)| \leqq 1. \tag{7}$$

*The sign of equality holds in* (6) (*for* $r \neq 0$) *and in* (7), *if and only if* $|f'(z)| \equiv 1$; *that is, if and only if the transformation* $w = f(z)$ *is a rigid motion.*

Let the real functions

$$x_j = x_j(z) = x_j(u + iv), \qquad j = 1, 2, 3, \tag{8}$$

have continuous derivatives of the third order with respect to $u$, $v$, in $|z| < 1$, with

$$E = G = [\lambda(z)]^2, \qquad F = 0, \tag{9}$$

where $E$, $F$, $G$ are the fundamental quantities of the first order, so that the functions (8) give an isothermic map of $|z| < 1$ on a surface $S$, that is, a map which is conformal except at points where $\lambda = 0$.

The Gaussian curvature $K$ is defined on $S$ except at points where $\lambda = 0$.

The function

$$l(r, \theta; \lambda) = \int_0^r \lambda(\rho e^{i\theta}) d\rho$$

denotes now the length of the image on $S$ of the segment between the points $z=0$ and $z=re^{i\theta}$. We have the following further special case of Theorem 2:

THEOREM 4. *Let the Gaussian curvature $K$ be less than or equal to zero wherever $K$ is defined on the above surface $S$. If*

$$l(r, \theta; \lambda) \leqq 1$$

*for all $(r, \theta)$ with $r<1$, then*

$$l(r, \theta; \lambda) \leqq r \tag{10}$$

*and*

$$\lambda(0) \leqq 1. \tag{11}$$

*The sign of equality holds in* (10) (*for $r\neq 0$*) *and in* (11), *if and only if $\lambda(z)\equiv 1$; that is, if and only if $S$ is a developable piece of surface and is a geodesic circle given in isometric representation.*

Theorem 3 is a special case of Theorem 4.

A generalization of Theorem 1 to isothermic maps on minimal surfaces, similar to the above generalization of Theorem 3 to isothermic maps on surfaces of non-positive curvature, previously has been given.* Since $K\leqq 0$ on minimal surfaces, Theorem 4 applies in particular to minimal surfaces.

An easily obtained generalization of Liouville's theorem is that if $F(z)=F(u+iv)$ is a harmonic function of $u$ and $v$ in the entire finite plane and is bounded, then $F(z)$ is identically constant. It follows that if the functions (8), not necessarily satisfying (9), are harmonic in the entire finite plane, and if the corresponding surface $S$ is bounded, then $S$ reduces to a point. In particular, since the coordinate functions of a minimal surface in isothermic representation necessarily are harmonic, *if the functions* (8) *give an isothermic representation of the entire finite plane on a minimal surface $S$, and if $S$ is bounded, then $S$ reduces to a point.*

From Theorem 4 we shall obtain the following similar generalization to space of Liouville's theorem:

THEOREM 5. *Let the functions* (8) *satisfy* (9) *in the entire finite plane, and let the Gaussian curvature $K$ be less than or equal to zero wherever $K$ is defined on the corresponding surface $S$. If $l(r, \theta; \lambda)$ is bounded,*

$$l(r, \theta; \lambda) \leqq M,$$

*for all $(r, \theta)$, then $S$ reduces to a point.*

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* E. F. Beckenbach and T. Radó, *Subharmonic functions and minimal surfaces*, Transactions of this Society, vol. 35 (1933), pp. 648–661.

Finally, we shall point out certain forms which our results show to be positive definite and shall add a further algebraic proof that these forms are positive definite.

2. **Subharmonic functions and functions of class** PL. In this section we present the definitions of subharmonic functions and functions of class PL and list the results concerning these functions which we shall use in the sequel.*

2.1. Let $g(z)$ be defined in a domain $D$ (connected open set), and assume $-\infty \leqq g(z) < +\infty$ in $D$. Suppose that $g(z)$ is not identically equal to $-\infty$ in $D$, that $g(z)$ is upper semi-continuous in $D$, and that, for every domain $D'$ lying together with its boundary $B'$ in $D$ and for every function $h(z)$ continuous in $D'+B'$, harmonic in $D'$, and satisfying $h(z) \geqq g(z)$ on $B'$, we have $h(z) \geqq g(z)$ in $D'$. Then $g(z)$ is said to be *subharmonic* in $D$.

2.2. A function $p(z)$, defined in a domain $D$, is said to be *of class* PL in $D$ provided that $p(z) \geqq 0$ in $D$ and $\log p(z)$ is subharmonic. It is understood that $\log p(z) = -\infty$ at points where $p(z) = 0$.

2.3. If $p(z)$ is of class PL in $D$, then $p(z)$ is subharmonic in $D$.

2.4. If a real function $g(z)$ admits continuous second derivatives, then a necessary and sufficient condition that $g(z)$ be subharmonic is that its Laplacian be non-negative:

$$\Delta g = g_{uu} + g_{vv} \geqq 0.$$

2.5. A subharmonic function $g(z)$ cannot attain its least upper bound at any (interior) point of its domain of definition $D$, unless $g(z)$ is identically constant.

2.6. If $g(z)$ is subharmonic in a domain $D$, and if $\Gamma$ is a smooth Jordan curve in $D$, then $g(z)$ is summable on $\Gamma$ as a function of the arc length.

2.7. The sum of a finite number of functions, subharmonic (or of class PL) in a domain $D$, is again a function subharmonic (or of class PL) in $D$.

2.8. The product of a finite number of functions of class PL in a domain $D$ is again a function of class PL in $D$.

2.9. If the function $g(z)$ is subharmonic (or of class PL) in a domain $D$, then the sequence

$$g(z; k) = \frac{k^2}{\pi} \iint_{\rho<1/k} g(z + \rho e^{i\theta}) \rho d\rho d\theta, \quad k = 1, 2, \cdots,$$

* For the definitions and results of this section, and for references to their sources see T. Radó, *Subharmonic Functions*, Springer, Berlin, 1937, particularly chaps. 1–3.

has the following properties. Let $D'$ be a domain lying together with its boundary in $D$. Then for large $k$ the function $g(z;\, k)$ is defined, subharmonic (or of class PL), and continuous in $D'$; and

$$g(z;\, k) \to g(z) \tag{12}$$

with

$$g(z;\, k) \geqq g(z;\, k+1) \tag{13}$$

in $D'$. We indicate that (12) and (13) both hold by writing $g(z;\, k) \searrow g(z)$.

2.10. Let there be given a domain $D$ and a sequence of functions $\{g_n(z)\}$ such that if $D'$ is any domain lying together with its boundary in $D$, then for large $n$ the functions $g_n(z), g_{n+1}(z), \cdots$ are defined and subharmonic (or of class PL) in $D'$. If $\{g_n(z)\}$ converges uniformly in $D'$ to a function $g(z)$, then $g(z)$ is subharmonic (or is either of class PL or identically zero) in $D$.

2.11. If, instead of converging uniformly in $D'$, the sequence $\{g_n(z)\}$ of 2.10 satisfies

$$g_n(z) \geqq g_{n+1}(z)$$

in $D'$, then $\{g_n(z)\}$ converges either to a subharmonic function or to $-\infty$ (or converges either to a function of class PL or to zero) throughout $D$.

3. **Lemma.** Of the three related results of this section, only 3.1, restricted to the easily proved special case in which $g(z)$ is continuous, will be used as a lemma in proving Theorem 2. The others are included for the sake of completeness.

3.1. *If $g(z)$ is subharmonic in $|z|<1$, then the function*

$$h(re^{i\theta}) = \frac{1}{r}\int_0^r g(\rho e^{i\theta})d\rho, \qquad r \neq 0, \tag{14}$$
$$h(0) = g(0),$$

*is again subharmonic in $r<1$.*

Proof. Suppose first that $g(z)$ is continuous. For any positive integer $n$, the function

$$h_n(re^{i\theta}) = \sum_{l=1}^{n} g(lre^{i\theta}/n)\,\frac{1}{n}$$

is subharmonic in $r<1$ by 2.7. Further, the sequence $\{h_n(re^{i\theta})\}$ converges uniformly to $h(re^{i\theta})$ in any closed region in $r<1$; hence, by 2.10 the function

$$h(re^{i\theta}) = \lim_{n\to\infty} h_n(re^{i\theta})$$

is subharmonic in $r<1$.

If $g(z)$ is a general subharmonic function, we can no longer consider $g(z)$ as a limit of Riemann sums, but may proceed as follows. The function $g(z; k)$, defined in 2.9, is defined, subharmonic, and continuous in $|z|<1-1/k$. Hence by the analysis of the preceding paragraph, the function

$$h(re^{i\theta}; k) = \frac{1}{r}\int_0^r g(\rho e^{i\theta}; k)d\rho$$

is subharmonic in $r<1-1/k$. Further, by 2.9,

$$g(\rho e^{i\theta}; k) \searrow g(\rho e^{i\theta}),$$

and by 2.6 the integral (14) exists; hence

$$\int_0^r g(\rho e^{i\theta}; k)d\rho \geqq \int_0^r g(\rho e^{i\theta})d\rho,$$

and*

$$\lim_{k\to\infty}\int_0^r g(\rho e^{i\theta}; k)d\rho = \int_0^r g(\rho e^{i\theta})d\rho;$$

that is,

$$h(re^{i\theta}; k) \searrow h(re^{i\theta}).$$

Therefore, by 2.11, $h(re^{i\theta})$ is subharmonic.

Similarly, we have the following result:

3.2. *If $p(z)$ is of class* PL *in $|z|<1$, then the function*

$$q(re^{i\theta}) = \frac{1}{r}\int_0^r p(\rho e^{i\theta})d\rho, \qquad r \neq 0,$$
$$q(0) = p(0),$$

*is again of class* PL *in $r<1$.*

3.3. *If $p(z)$ is of class* PL *in $|z|<1$, then the function*

$$t(re^{i\theta}) = \int_0^r p(\rho e^{i\theta})d\rho$$

*is again of class* PL *in $r<1$.*

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* See for instance S. Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 63.

PROOF. Since

$$t(re^{i\theta}) = rq(re^{i\theta}),$$

our result follows from 2.8, 3.2, and the fact that the function $r=|z|$ is of class PL.

However, the hypothesis that $g(z)$ is subharmonic for $|z|<1$ does not imply that the function

$$k(re^{i\theta}) = \int_0^r g(\rho e^{i\theta})d\rho$$

is subharmonic for $r<1$. For example, the function

$$g(z) = R(z) + 1 = u + 1$$

is (positive and) subharmonic for $|z|<1$. But for the corresponding function

$$k(re^{i\theta}) = \frac{r^2}{2} \cos\theta + r,$$

we have

$$\Delta k(re^{i\theta}) = \frac{3}{2} \cos\theta + \frac{1}{r};$$

in particular,

$$\Delta k\left(\frac{3}{4} e^{i\pi}\right) = -\frac{3}{2} + \frac{4}{3} = -\frac{1}{6} < 0,$$

so that, by 2.4, $k(re^{i\theta})$ is not subharmonic throughout $r<1$.

4. **Proof of Theorem** 2. Under the hypotheses of Theorem 2, the function

$$h(re^{i\theta}) = \frac{1}{r} l(r, \theta; g), \qquad r \neq 0,$$

$$h(0) = g(0),$$

is subharmonic for $r<1$ by 3.1. From (3) we have

$$\limsup_{\rho\to 1,\ \rho<1} h(\rho e^{i\theta}) \leqq 1;$$

whence by 2.5

$$h(re^{i\theta}) \leqq 1, \tag{15}$$

the sign of equality holding either throughout the circle $r<1$ or nowhere in $r<1$. Now (4) and (5) follow from (15) and the definition of $h(re^{i\theta})$.

5. **On Theorems** 3 **and** 4. If $f(z)$ is analytic, then $|f(z)|$ is of class PL. Indeed, if $f(z)\neq 0$ in a domain $D$, then $\log |f(z)|$, which is the real part of the analytic function $\log f(z)$, is harmonic there, and hence, by 2.4, is subharmonic.

Again, if for the functions (8) the equations (9) are satisfied, then the Gaussian curvature $K$ of the corresponding surface $S$ is given, wherever it is defined, by

$$K = -\frac{1}{\lambda}\Delta \log \lambda.*$$

Hence, by 2.4, $K\leqq 0$ wherever $K$ is defined on $S$ if and only if $\lambda(z)$ is of class PL in $|z|<1$.

Since $|f'(z)|$ and $\lambda(z)$ are of class PL, they are subharmonic by 2.3, so that Theorems 3 and 4 follow as particular cases of Theorem 2. Indeed, Theorem 3 is a particular case of Theorem 4, with $K\equiv 0$.

Though $|f'(z)|$ and $\lambda(z)$ are of class PL, we have used only the weaker condition that they are subharmonic. There actually exist conformal maps on surfaces $S$ with $K>0$ for which $\lambda(z)$ is subharmonic, and for these maps Theorem 4 still holds. However, $\lambda(z)$ is subharmonic for all isothermic maps of $z$ domains on a surface $S$ if and only if $K\leqq 0$ wherever $K$ is defined on $S$;† so the condition that $\lambda(z)$ is subharmonic can be used for the surface $S$ irrespective of the isothermic parameters if and only if $K\leqq 0$ wherever $K$ is defined on $S$.

6. **Proof of Theorem** 5. Let the functions (8) satisfy the hypotheses of Theorem 5, and let $Z=z/R$, where $R$ is positive. For $|z|=r<R$, the functions

$$X_j(Z) = x_j(z)/M, \qquad j = 1, 2, 3,$$

as functions of $Z$, satisfy the conditions of Theorem 4. Hence, for $r<R$

$$l(r, \theta; \lambda) < M|Z| = Mr/R.$$

Keeping $z=re^{i\theta}$ constant and letting $R$ tend to $\infty$, we obtain

$$l(r, \theta; \lambda) = 0.$$

* See E. F. Beckenbach and T. Radó, *Subharmonic functions and surfaces of negative curvature*, Transactions of this Society, vol. 35 (1933), pp. 662–674.

† E. F. Beckenbach, *On subharmonic functions*, Duke Mathematical Journal, vol. 1 (1935), pp. 480–483.

7. **On positive definite forms.** If the map of $|z|<1$ given by $w=f(z)$ is everywhere conformal, so that $f'(z)\neq 0$, then we may write

$$f'(z) = [\phi(z)]^2,$$

where $\phi(z)$ is analytic in $|z|<1$. Let

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n;$$

then

$$h(r, \theta; |f'|) = \frac{1}{r} l(r, \theta; \phi\bar{\phi})$$
$$= \sum_{n,m=0}^{\infty} \frac{a_n \bar{a}_m}{n+m+1} r^{n+m} e^{(n-m)i\theta},$$

where $\bar{\alpha}$ denotes the complex number conjugate to $\alpha$.

Direct computation yields

$$\Delta h(r, \theta; \phi\bar{\phi}) = \sum_{n,m=1}^{\infty} a_n \bar{a}_m r^{n+m-2} e^{(n-m)i\theta} \frac{4nm}{n+m+1}$$
$$(16) \qquad = \sum_{n,m=1}^{\infty} \frac{b_n \bar{b}_m}{n+m+1},$$

where $b_n = 2na_n r^{n-1} e^{ni\theta}$.

Since, by 3.1, $h(r, \theta; \phi\bar{\phi})$ is subharmonic, the expression (16) is non-negative by 2.4. On the other hand, we shall show directly that (16) is non-negative and thus obtain an alternative proof of Lemma 3.1 for the special case under consideration.

That the Hermitian form (16) is positive definite follows from the identity

$$\sum_{n,m=1}^{\infty} \frac{b_n \bar{b}_m}{n+m+1} = \int_0^1 \left| \sum_{n=1}^{\infty} b_n \rho^n \right|^2 d\rho$$
$$= \frac{4}{r^3} \int_0^r |\phi'(\rho e^{i\theta})|^2 \rho d\rho,$$

or equally well from the identity*

$$\sum_{n,m=1}^{\infty} \frac{b_n \bar{b}_m}{n+m+1} = \sum_{j=0}^{\infty} \left| \sum_{n=1}^{\infty} \frac{n^j}{(n+1)^{j+1}} b_n \right|^2.$$

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* Suggested by Professor H. E. Bray.

The stronger result of 3.3 shows that $l(r, \theta; \phi\bar{\phi})$ is of class PL; whence

$$(17)\quad \begin{aligned}&[l(r, \theta; \phi\bar{\phi})]^2\Delta \log l(r, \theta; \phi\bar{\phi})\\ &= \sum_{g,h,j,k=1}^{\infty} \frac{a_g\bar{a}_h a_j\bar{a}_k(g-h)^2(j-k)^2}{(g+h+1)(j+k+1)(g+k+1)(h+j+1)}\\ &\quad\cdot r^{g+h+j+k}e^{i(g-h+j-k)} \geqq 0.\end{aligned}$$

On the other hand, it can be shown directly, by an extension of the above identities, that (17) is positive definite.

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# SOME ITERATED INTEGRALS IN THE FRACTIONAL CALCULUS

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1. **Introduction.** A considerable amount of attention has been devoted to integrals of fractional order, both in regard to their applications and to the conditions for their existence.* We shall denote the fractional integral of order $\alpha$ by

$$(1)\qquad {}_TI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_T^t (t-v)^{\alpha-1}f(v)dv, \qquad \alpha > 0, t > T,$$

and it is the purpose of this paper to give some formulas which may be of use in manipulating these integrals. We shall prove that under certain conditions the following relations hold:

$$(2)\qquad \int_T^\infty \frac{{}_TI_t^\alpha f(t)}{t^{k+\alpha}}dt = \frac{\Gamma(k)}{\Gamma(k+\alpha)}\int_T^\infty \frac{f(t)}{t^k}dt, \qquad \alpha > 0,$$

$$(3)\qquad \int_T^\infty e^{-kt}{}_TI_t^\alpha f(t)dt = k^{-\alpha}\int_T^\infty e^{-kt}f(t)dt, \qquad \alpha > 0,$$

$$(4)\qquad \int_T^\infty \cos kt\, {}_TI_t^\alpha f(t)dt = k^{-\alpha}\int_T^\infty \cos(kt + \pi\alpha/2)f(t)dt, \quad 0 < \alpha < 1,$$

and (4) holds when cosine is replaced by sine. As an application we

* A bibliography is given by H. T. Davis, *Application of fractional operators to functional equations*, American Journal of Mathematics, vol. 49 (1927), pp. 123–142. See also J. D. Tamarkin, *On integrable solutions of Abel's integral equation*, Annals of Mathematics, (2), vol. 31 (1930), pp. 219–229.