APOLARITY OF TRILINEAR FORMS AND PENCILS OF BILINEAR FORMS*

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1. **Introduction.** In this note the idea of apolar trilinear forms, introduced earlier by the author,† is generalized, and the main theorems are given new proofs independent of the group theoretic methods and notation used then. The importance of the apolarity concept in trilinear form classification is illustrated by a numerical application. In §3 there is given a new method of classifying singular pencils of bilinear forms based on Dickson's minimal numbers.‡

2. **Apolarity of trilinear forms.** A trilinear form $F(x, y, z)$ $=\sum a_{hij}x_h y_i z_j$, where h runs from 1 to l, i from 1 to m, and j from 1 to n, has as *two-way rank invariants*§ r_h, r_i, r_j the smallest numbers of variables x, y, z, respectively, in terms of which the form can be expressed. Let F and $F'=\sum b_{hij}x_h y_i z_j$ be two trilinear forms in which the numbers of x's, y's, z's are r_h, m, n and r_h', m, n, respectively, where r_h and r_h' are the two-way h-rank invariants of F and F'.

DEFINITION. *F and F' are said to be h apolar (relative to m, n)‖ if*

$$(1) \qquad \sum_{i,j=1}^{m,n} a_{hij}b_{\nu ij} = 0, \qquad h = 1, \cdots, r_h; \nu = 1, \cdots, r_h',$$

and

$$(2) \qquad\qquad r_h + r_h' = mn.$$

We define i and j apolarity analogously with respect to the i and j two-way rank invariants of F and F'. We shall apply the term *apolar* to two forms if they are h, i, or j apolar. The following theorems concerning h apolarity can, of course, be rephrased in terms of i and j apolarity.

We consider questions of existence and uniqueness properties of

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forms F' apolar to a given form F (relative to m, n). Let $a_{hij} = a_{h\lambda}$, where $\lambda = (i-1)n + j$.* Defining $b_{\nu ij} = b_{\nu\lambda}$, we can write (1) as

$$(1')\qquad \sum_{\lambda=1}^{mn} a_{h\lambda}b_{\nu\lambda} = 0, \qquad h = 1, \cdots, r_h; \nu = 1, \cdots, r_h = mn - r_h.$$

The rank of $(a_{h\lambda})$ is r_h, for if it were less, F could be expressed in terms of less than r_h variables x. The $b_{\nu\lambda}$ in A' can be interpreted as a set of r_h' linearly independent solutions of the equations $\sum_{\lambda=1}^{mn} a_{h\lambda}t_\lambda = 0$, $(h = 1, \cdots, r_h)$. It is well known that such solutions $b_{\nu\lambda}$ always exist, and that furthermore, if $c_{\mu\lambda}$ is a second set of solutions, constants $k_{\mu\nu}$ exist such that

$$(3')\qquad\qquad c_{\mu\lambda} = \sum_{\nu=1}^{r_h'} k_{\mu\nu}b_{\nu\lambda}, \qquad\qquad \mu = 1, \cdots, r_h',$$

where $|k_{\mu\nu}| \neq 0$.† Formula (1) is the usual definition of apolarity of two-way matrices. If we write $c_{\mu ij} = c_{\mu\lambda}$, (3') becomes

$$(3)\qquad\qquad c_{\mu ij} = \sum_{\nu=1}^{r_h'} k_{\mu\nu}b_{\nu ij}, \qquad\qquad \mu = 1, \cdots, r_h'.$$

If we subject the x_ν in F' to the non-singular linear transformation $x_\nu = \sum_\mu k_{\mu\nu}x_\mu'$, F' becomes

$$\sum_{\nu,i,j} b_{\nu ij}\left(\sum_\mu k_{\mu\nu}x_\mu'\right)y_i z_j = \sum_{\mu,i,j}\left(\sum_\nu k_{\mu\nu}b_{\nu ij}\right)x_\mu' y_i z_j = \sum_{\mu,i,j} c_{\mu ij}x_\mu' y_i z_j.$$

We have proved the following theorem:

THEOREM 1. *For every trilinear form F with invariants r_h, r_i, r_j and every pair of integers m ($\geq r_i$), n ($\geq r_j$) there exists a trilinear form F' which is h apolar to F. Any second form F'' which is h apolar to F (relative to m, n) is conjugate to F' under non-singular linear transformations on the variables x_ν of F'.*

Since the steps are all rational, the coefficients of F' can be chosen from the field of the coefficients of F.

Two trilinear forms F and G are said to be *equivalent*, written $F \sim G$, if F can be sent into G by non-singular linear transformations on its sets of variables x, y, z, taken separately.

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* The matrix $(a_{h\lambda})$ is called the *two-way h-display* of F. See R. Oldenburger, *Non-singular multilinear forms and certain p-way matrix factorizations*, Transactions of this Society, vol. 39 (1936), pp. 423–424.

† See, for example, J. H. M. Wedderburn, *Lectures on Matrices*, American Mathematical Society Colloquium Publications, vol. 17, New York, 1934, p. 19, Theorem 10.

THEOREM 2. *If $F \sim G$, and if F', G' are h apolar to F, G, respectively, (relative to the same numbers m, n), then $F' \sim G'$.*

If F can be sent into G by transformations on x alone, then F is, by Theorem 1, h apolar to G'; whence $F' \sim G'$. Next suppose that $G = \sum c_{h\mu j} x_h y'_\mu z_j$ is obtained from F by the transformation T, $y_i = \sum k_{\mu i} y'_\mu$; that is, $c_{h\mu j} = \sum_i k_{\mu i} a_{hij}$. Then let $F^* = \sum d_{\nu\mu j} x_\nu y'_\mu z_j$ be obtained from $F' = \sum b_{\nu\rho j} x_\nu y_\rho z_j$ by applying the transformation $y_\rho = \sum \kappa_{\mu\rho} y'_\mu$, contragredient to T, that is, $\sum \kappa_{\mu\rho} k_{\mu i} = \delta_{\rho i}$ (Kronecker's δ_{ij}). We see that F^* is h apolar to G, for

$$\sum_{\mu, j} d_{\nu\mu j} c_{h\mu j} = \sum_{i, \rho, j} \left(\sum_\mu \kappa_{\mu\rho} k_{\mu i} \right) b_{\nu\rho j} a_{hij} = \sum_{i, \rho, j} \delta_{i\rho} b_{\nu\rho j} a_{hij},$$

$$= \sum_{i, j} b_{\nu i j} a_{h i j} = 0,$$

since F' and F are h apolar. Then $F' \sim F^* \sim G'$ gives $F' \sim G'$, where G' is any form h apolar to G. By symmetry the same result holds for linear transformations on z_j. The remainder of the argument is obvious and will be omitted.

Theorem 2 enables us to speak of apolar classes of forms $[F]$ and $[F']$. ($[F]$ denotes the class of forms equivalent to F.) Theorems 1 and 2, together with their analogs for i and j apolarity, can be applied to reduce the problem of classifying all trilinear forms to that of classifying trilinear forms in which no one of the rank invariants r_h, r_i, r_j is greater than half the product of the other two. To prove this we note first that no one of the r_ν can exceed the product of the other two.† Then it is obvious that one of r_h and $r'_h = r_i r_j - r_h$ does not exceed $r_i r_j / 2$.

An example will serve to illuminate such application. Let F have $r_h = 27$, $r_i = 10$, $r_j = 3$. Then if we take $m = 10$, $n = 3$, $[F]$ is h apolar to a unique class of forms $[F']$ with $r'_h = 3$, $r'_i \le 10$, $r'_j \le 3$. As $[F]$ runs through the classes of forms $(27, 10, 3)$, there will be classes $[F']$ for each value of $r'_j \le 3$ and $r'_h / r'_j = 3 / r'_j \le r'_i \le 3 r'_j = r'_i r'_j$. Hence, given the classes of forms with $(r_h, r_i, r_j) = (3, 9, 3)$, $(3, 8, 3)$, $\cdots$, $(3, 1, 3)$, $(3, 6, 2)$, $(3, 5, 2)$, $\cdots$, $(3, 2, 2)$, $(3, 3, 1)$, we can at once write down the classes of forms $(27, 10, 3)$.

But in the same way, the forms $(3, 9, 3)$, $\cdots$, $(3, 5, 3)$ are i apolar to forms with $r_h \le 3$, $r_i \le 4$, $r_j \le 3$; and the forms $(2, 6, 3)$, $(2, 5, 3)$, $(2, 4, 3)$ are apolar to forms with the r_ν all ≤ 3. Hence the classification of forms $(27, 10, 3)$ with $810 = 3 \cdot 10 \cdot 27$ coefficients can be re-

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† H. R. Brahana, *Metabelian groups and trilinear forms*, Duke Mathematical Journal, vol. 1 (1935), p. 191.

duced by apolarity considerations to the solution of linear equations and to the classification of forms in which all the r_ν are less than or equal to three or $(r_h, r_i, r_j) = (3, 4, 3)$ and in which at most 36 coefficients appear.

3. **Pencils of bilinear forms.** Consider the trilinear form

$$G = \sum_{\nu=1}^{R} R_\nu + \sum_{\lambda=1}^{S} S_\lambda + T,$$

where

$$T = z_1 \sum_{\alpha,\beta=1}^{n} t_{\alpha\beta} x_\alpha y_\beta + z_2 \sum_{\alpha=1}^{n} x_\alpha y_\alpha, \quad R_\nu = \sum_{\alpha=1}^{m_\nu} (z_1 x_{\nu\alpha-1} + z_2 x_{\nu\alpha}) y_{\nu\alpha},$$

$$S_\lambda = \sum_{\alpha=1}^{n_\lambda} X_{\lambda\alpha}(z_1 Y_{\lambda\alpha-1} + z_2 Y_{\lambda\alpha}),$$

and all of the variables in the forms R_ν, S_λ, T are linearly independent in the field K of G. We suppose also that $m_\nu \geqq m_{\nu+1}$, $n_\lambda \geqq n_{\lambda+1}$. The form G has two-way rank invariants

$$r_h = \sum_{\nu=1}^{R} (m_\nu + 1) + \sum_{\lambda=1}^{S} n_\lambda + n, \quad r_i = \sum_{\nu=1}^{R} m_\nu + \sum_{\lambda=1}^{S} (n_\lambda + 1) + n,$$

$$r_j = 2.$$

The form $\overline{G} = \sum_{\nu=1}^{\overline{R}} \overline{R}_\nu + \sum_{\lambda=1}^{\overline{S}} \overline{S}_\lambda + \overline{T}$ is h apolar to G (relative to r_i and 2),† where

$$\overline{T} = z_1 \sum_{\alpha=1}^{n} x_\alpha y_\alpha - z_2 \sum_{\alpha,\beta=1}^{n} t_{\beta\alpha} x_\alpha y_\beta,$$

$$\overline{R}_\nu = \sum_{\alpha=1}^{m_\nu-1} x_{\nu\alpha}(z_1 y_{\nu\alpha+1} - z_2 y_{.\alpha}),$$

$$\overline{S}_\lambda = \sum_{\alpha=1}^{n_\lambda+1} Y_{\lambda\alpha-1}(z_1 X_{\lambda\alpha} - z_2 X_{\lambda\alpha-1}),$$

and the variables x and y are all independent. We have $\overline{S} = S$, $R - \overline{R} = j_1$, where j_α is the number of m_ν equal to α.

If in G we let $z_1' = -z_2$, $z_2' = z_1$, $x_\alpha = y_\alpha'$, $x_{\nu\alpha} = y_{\nu\alpha}'$, $X_{\lambda\alpha} = Y_{\lambda\alpha}'$, $y_\alpha = x_\alpha'$, $y_{\nu\alpha} = x_{\nu\alpha-1}'$, $Y_{\lambda\alpha} = X_{\lambda\alpha-1}$, we obtain a form G' identical in form with G, having $T'(x, y) \equiv T(x, y)$ and the numbers $m_\nu' = m_\nu - 1$, $n_\lambda' = n_\lambda + 1$.

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† That condition (1) for apolarity is satisfied is obvious if the two-way h-displays of G and $\overline{G}$ are written down. For (2) we note that $\bar{r}_h + r_h = (\sum(m_\nu - 1) + \sum(n_\lambda + 2) + n) + (\sum(m_\nu + 1) + \sum n_\lambda + n) = 2(\sum m_\nu + \sum(n_\lambda + 1) + n) = 2r_i$.

(We omit R_ν' if $m_\nu' = 0$.) Then $S' = S$, $R' = R - j_1$. Hence G' has two-way rank invariants

$$r_h' = \sum_{\nu=1}^{R'} m_\nu + \sum_{\lambda=1}^{S} (n_\lambda + 1) + n,$$

$$r_i' = \sum_{\nu=1}^{R'} (m_\nu - 1) + \sum_{\lambda=1}^{S} (n_\lambda + 2) + n.$$

Considered as a pencil of bilinear forms with parameter $z = (z_1, z_2)$, G has minimal numbers m_ν and n_λ and invariant factors (aside from 1) equal to those of T. Dickson* has proved that any pencil of bilinear forms F can be put in the canonical form G by non-singular (rational) transformations on the variables x, y, z. We have then $F \sim G$. Let $\overline{F}$ be h apolar to F (relative to r_i and 2). Then, by Theorem 2, $\overline{F} \sim \overline{G}$; and if F' is for $\overline{F}$ the analogue† of G' for $\overline{G}$, then $F' \sim G'$. This means in particular that F' has the same minimal numbers as G' and invariant factors projectively equivalent to those of F (since F, G, G', F' have, in turn, projectively equivalent invariant factors).

Let $F^{(\alpha)}$ be derived from $F^{(\alpha-1)}$ as F' is derived from $F^{(0)} = F$. We propose to calculate the minimal numbers m_ν from the sequence of two-way rank invariants $r_h^{(\alpha)}$, $r_i^{(\alpha)}$. Since these invariants are ranks of two-way matrices, we will then have the m_ν expressed in terms of ranks of matrices, obtainable from F by solving linear equations.

To effect this calculation we shall evaluate the j_α defined above. By hypothesis m_1 is the biggest of the m_α, hence $j_\alpha = 0$ for $\alpha > m_1$. Hence, as soon as we know the sequence j_α, $(\alpha = 1, \cdots, m_1)$, we know all of the minimal numbers m_ν. In particular $R^{(0)} = R = \sum_{\alpha=1}^{m_1} j_\alpha$, and $\sum_{\nu=1}^{R} m_\nu = \sum_{\alpha=1}^{m_1} \alpha j_\alpha$.

We saw above that $j_1 = R^{(0)} - R^{(1)}$. Now

$$r_h^{(0)} - r_h^{(1)} = \left(\sum_{\nu=1}^{R^{(0)}} (m_\nu + 1) + \sum_{\lambda=1}^{S^{(0)}} n_\lambda + n \right)$$

$$- \left(\sum_{\nu=1}^{R^{(0)}} m_\nu + \sum_{\lambda=1}^{S^{(0)}} (n_\lambda + 1) + n \right)$$

$$= [R^{(0)} + (R^{(0)} - R^{(1)}) - S^{(0)},$$

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* Loc. cit., in particular Theorem 2, p. 248 and Theorem 3, p. 251. These results are stated for pairs of bilinear forms, but the numbers m_ν, n_λ and the projective invariants of the invariant factors will constitute a complete set of invariants for pencils of bilinear forms. We use here this generalization of Dickson's result.

† That is, the matrix of F' is the transpose of that of F.

and similarly $r_i{}^{(0)} - r_i{}^{(1)} = R^{(0)} - S^{(0)}$. Hence $j_1 = R^{(0)} - R^{(1)} = (r_h{}^{(0)} - r_h{}^{(1)}) - (r_i{}^{(0)} - r_i{}^{(1)}) = (r_h{}^{(0)} - r_i{}^{(0)}) - (r_h{}^{(1)} - r_i{}^{(1)})$. By induction on α we show that $j_\alpha = (r_h{}^{(\alpha-1)} - r_i{}^{(\alpha-1)}) - (r_h{}^{(\alpha)} - r_i{}^{(\alpha)})$, which is the desired result.

The analogous process when we use i apolarity gives the numbers n_λ.

The following inequality gives an upper limit for m_1 and is of use in determining the number of steps the above procedure will require. If $M < m_1$, then $R > \sum_{\alpha=1}^{M} j_\alpha$ and

$$\sum_{\alpha=1}^{M} \alpha j_\alpha + (M+1) \leq \sum_{\alpha=1}^{M} \alpha j_\alpha + m_1 \leq \sum_{\alpha=1}^{m_1} \alpha j_\alpha = \sum_{\nu=1}^{R} m_\nu.$$

But

$$\sum_{\nu=1}^{R} m_\nu = r_h - n - \sum_{\lambda=1}^{S} n_\lambda - R \leq r_h - n - R - S \leq 2r_h - r_i - n - 2R.$$

$(S = R + r_i - r_h.)$ Hence $M < 2r_h - r_i - n - \sum_{\alpha=1}^{M}(\alpha+2)j_\alpha - 2$. Therefore m_1 is less than the smallest integer M for which this inequality does not hold. Or, if one computes the rank, $r_h - R = r_i - S$, of the characteristic matrix $M_z(z)$ of F, the value of R thus obtained, together with $R = \sum_{\alpha=1}^{m_1} j_\alpha$, shows one exactly when enough steps have been taken.

Dickson's process* for determining m_ν, n_λ requires $R + S$ steps, each of which involves solving several sets of linear equations, and finding the minimum degree $m_{R-\alpha}$ in z of the forms $A_h{}^{(\alpha)}$ such that $\sum A_h{}^{(\alpha)} F_h{}^{(\alpha)} \equiv 0$ in z and y, where $F_h{}^{(\alpha)}$ is the coefficient of x_h in $F^{(\alpha)}$ (the form obtained by the $(\alpha-1)$st step). No method, other than trial and error, is given for determining $m_{R-\alpha}$.

The apolarity process requires $m_1 + n_1$ steps, each of which involves solving one set of linear equations and evaluating the rank of a matrix (which is also the first step in solving the next set of linear equations). This gives, at the αth step, a specific formula for the number of m_ν equal to α (which is the chief value of the method).

Hence, if $R + S$ is small, Dickson's method is the shorter, but if $m_1 + n_1 < R + S$, or if $R + S$ and $m_1 + n_1$ are large, the apolarity method is shorter.

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* Loc. cit.