A SUFFICIENCY PROOF FOR ISOPERIMETRIC PROBLEMS IN THE CALCULUS OF VARIATIONS

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The purpose of the present paper is to show that the sufficiency theorems for a strong relative minimum for isoperimetric problems can be obtained from those for simple integral problems with only a little additional argument. The method here used is a simple extension of one used by Birkhoff and Hestenes* for a special isoperimetric problem. Heretofore sufficiency theorems of this type have been obtained from those for a restricted relative minimum by the application of the theorem of Lindeberg. Sufficiency theorems for a restricted relative minimum can be obtained either from the theories of the problems of Lagrange and Bolza or by an argument analogous to that used for simple integrals.

The problem to be considered is that of minimizing an integral

$$J = \int_{x_1}^{x_2} f(x, y, y')dx = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx$$

in a class of admissible arcs

$$y_i(x), \qquad x_1 \leqq x \leqq x_2;\ i = 1, \cdots, n, \tag{1}$$

joining two fixed points 1 and 2 and satisfying a set of isoperimetric conditions

$$J_\alpha = \int_{x_1}^{x_2} f_\alpha(x, y, y')dx = l_\alpha, \qquad \alpha = 1, \cdots, m, \tag{2}$$

where the l's are constants. It will be assumed that the functions f, f_α are defined and have continuous derivatives of the first three orders in a region $\mathcal{R}$ of points (x, y, y'). The points of $\mathcal{R}$ will be called *admissible.* A continuous arc (1) that can be divided into a finite number of subarcs on each of which it has continuous derivatives will be called *admissible* if its elements (x, y, y') are all admissible.

Associated with the problem is an integral of the form

$$I_\lambda = \int_{x_1}^{x_2} F(x, y, y', \lambda)dx,$$

* *Natural isoperimetric conditions in the calculus of variations,* Duke Mathematical Journal, vol. 1 (1935), pp. 251–258. The method used in this paper was suggested by Professor Birkhoff.

where $F=f+\lambda_\alpha f_\alpha$* and the λ's are constant multipliers. It is in terms of this integral that the sufficiency conditions stated below are given. An admissible arc (1) and a set of constants λ_α having continuous second derivatives will be said to form an *isoperimetric extremal* if they satisfy the Euler-Lagrange equations

$$F_{y_i} - \frac{dF_{y_i'}}{dx} = 0. \tag{3}$$

Let E_0 be an isoperimetric extremal joining the points 1 and 2 and satisfying the conditions (2). It will be assumed that E_0 is normal, that is, that the equations $P_{i\alpha}a_\alpha=0$, where

$$P_{i\alpha} = f_{\alpha y_i} - \frac{df_{\alpha y_i'}}{dx}, \tag{4}$$

hold along E_0 only in case the constants a_α are all zero. Subarcs of E_0 need not be normal. We shall suppose that the extremal E_0 has the following further properties. At each element (x, y, y', λ) in a neighborhood of those belonging to E_0 the inequality

$$F(x, y, Y', \lambda) - F(x, y, y', \lambda) - (Y_i' - y_i')F_{y_i'}(x, y, y', \lambda) > 0$$

holds for every admissible set $(x, y, Y') \neq (x, y, y')$. Moreover along E_0 the inequality $F_{y_i'y_k'}\pi_i\pi_k>0$ holds for every set $(\pi)\neq(0)$. Finally

$$Q(\eta) = \int_{x_1}^{x_2} 2\omega(x, \eta, \eta')dx,$$

the second variation of I_λ along E_0, is positive for every non-null set of admissible variations $\eta_i(x)$, $(x_1 \leqq x \leqq x_2)$, vanishing at $x=x_1$ and $x=x_2$ and satisfying with E_0 the equations

$$L_\alpha(\eta) = \int_{x_1}^{x_2} \{f_{\alpha y_i}\eta_i + f_{\alpha y_i'}\eta_i'\}dx = 0. \tag{5}$$

An admissible variation $\eta_i(x)$, $(x_1 \leqq x \leqq x_2)$, is an arc having continuity properties like those of admissible arcs. The integrand 2ω appearing in $Q(\eta)$ is of the form $2\omega = F_{y_iy_k}\eta_i\eta_k + 2F_{y_iy_k'}\eta_i\eta_k' + F_{y_i'y_k'}\eta_i'\eta_k'$.

The theorem to be proved is the following:

THEOREM. *If E_0 has the properties described above, there is a neighborhood $\mathcal{J}$ of E_0 in xy-space such that the inequality $J(C) > J(E_0)$ holds for every admissible arc C in $\mathcal{J}$ joining* 1 *and* 2, *satisfying conditions* (2), *and not identical with E_0.*

* Repeated indices denote summation.

The proof of this result is based on two lemmas to be given below. In these lemmas we use the following definition of conjugate points. A value $x_3 \neq x_1$ is said to define a point 3 conjugate to 1 on E_0 relative to I_λ if there is a solution $\eta_i(x)$ of the equations

$$K_i(\eta) \equiv \omega_{\eta_i} - \frac{d\omega_{\eta_i'}}{dx} = 0 \tag{6}$$

having continuous second derivatives and vanishing at x_1 and x_3 but not identically zero on x_1x_3.

LEMMA 1. *If E_0 has on it no point 3 conjugate to 1 relative to I_λ, there exist neighborhoods $\mathcal{J}$ of E_0 in xy-space, N of the end values of E_0 in $(x_1y_1x_2y_2)$-space, and Λ of the multipliers λ_α belonging to E_0 such that for every set $(x_1y_1x_2y_2)$ in N and λ_α in Λ there is an isoperimetric extremal E_λ in $\mathcal{J}$ having $(x_1y_1x_2y_2)$ as its end values and λ_α as its multipliers. Moreover for these values of λ_α the inequality $I_\lambda(C) > I_\lambda(E_\lambda)$ holds for every admissible arc C in $\mathcal{J}$ joining the ends of E_λ and not identical with E_λ.*

This lemma is analogous to one given by Hahn and has been established by Birkhoff and Hestenes* following a method given by Bliss† in the proof of a similar theorem for the problem of Bolza. It should be noted that this lemma does not depend on the normality of E_0.

LEMMA 2. *Let $t_0 = x_1 < t_1 < \cdots < t_{q+1} = x_2$ be a set of values such that there are no pairs of conjugate points on E_0 relative to I_λ on any of the intervals $t_{r-1} \leqq x \leqq t_r$, $(r = 1, \cdots, q+1)$. There exists a qn parameter family of broken isoperimetric extremals*

$$y_i(x, b_{11}, \cdots, b_{nq}), \qquad \lambda_\alpha(b_{11}, \cdots, b_{nq}), \qquad x_1 \leqq x \leqq x_2, \tag{7}$$

containing E_0 for $b_{is} = b_{is0}$, satisfying the conditions (2), passing through the points 1, 2, and $(x, y_i) = (t_s, b_{is})$, $(s = 1, \cdots, q)$, and having no corners on the intervals $t_{r-1} < x < t_r$. The functions $y_i(x, b)$, $y_{ix}(x, b)$, $\lambda_\alpha(b)$ have continuous first and second derivatives for values (x, b) near those belonging to E_0 except possibly at the corner points. For the arc E_b of the family (7) determined by values $(b) \neq (b_0)$ in a sufficiently small neighborhood of $(b) = (b_0)$ one has $J(E_b) > J(E_0)$.

In order to establish this result let

$$Y_i(x, b_{11}, \cdots, b_{nq}, \lambda_1, \cdots, \lambda_m), \qquad \lambda_\alpha, \qquad x_1 \leqq x \leqq x_2, \tag{8}$$

* Loc. cit., pp. 253–254.

† *The problem of Bolza in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 267–270.

be a $(qn+m)$ parameter family of broken extremals containing E_0 for values $b_{is}=b_{is0}$, $\lambda_\alpha=\lambda_{\alpha 0}$, passing through the points 1, 2, and $(x, y_i)=(t_s, b_{is})$ and having no corners on the intervals $t_{r-1}<x<t_r$. Except at the corner points the functions $Y_i(x, b, \lambda)$, $Y_{ix}(x, b, \lambda)$ have continuous first and second derivatives for values (x, b, λ) near those belonging to E_0. The existence of a family of this type follows readily from the first part of Lemma 1 and existence theorems for differential equations of the form (3). When the functions $Y_i(x, b, \lambda)$ are substituted in the integrals (2) a set of functions $J_\alpha(b, \lambda)$ is obtained having continuous first and second derivatives for values (b, λ) near those on E_0. The functional determinant $|\partial J_\alpha/\partial\lambda_\beta|$, $(\alpha, \beta=1, \cdots, m)$, is different from zero when $(b, \lambda)=(b_0, \lambda_0)$, as will be seen in the next paragraph. The equations $J_\alpha(b, \lambda)=l_\alpha$ are satisfied by the values $(b, \lambda)=(b_0, \lambda_0)$ and hence have unique solutions $\lambda_\alpha(b)$, with $\lambda_\alpha(b_0)=\lambda_{\alpha 0}$, having continuous second derivatives near $(b)=(b_0)$. When the functions $\lambda_\alpha(b)$ are substituted for λ_α in the functions (8), a family (7) is obtained having the properties described in the lemma. The first part of the lemma is immediate. In order to prove the last part we note that by virtue of the identities

$$y_i(x_1, b) = y_{i1}, \qquad y_i(t_s, b) = b_{is}, \qquad y_i(x_2, b) = y_{i2},$$

where (x_1, y_1), (x_2, y_2) are the points 1 and 2, respectively, the variations

$$\delta y_i(x) = y_{ib_{js}}(x, b_0)db_{js}, \qquad j = 1, \cdots, n;\ s = 1, \cdots, q,$$

satisfy the relations

$$\delta y_i(x_1) = 0, \qquad \delta y_i(t_s) = db_{is}, \qquad \delta y_i(x_2) = 0,$$

so that $\delta y_i(x)\neq 0$ on x_1x_2 if $(db)\neq(0)$. Moreover the variations $\eta_i=\delta y_i$ satisfy equations (5). For by construction the value $J_\alpha(b)$ of the integral J_α along the family (7) is a constant. It follows that $dJ_\alpha=L_\alpha(\delta y)=0$ along E_0. From the assumptions on E_0 we have accordingly $Q(\delta y)>0$ whenever $(db)\neq(0)$. Consider now the function

$$J_0(b) = J(b) + \lambda_{\alpha 0}J_\alpha(b)$$

obtained by evaluating the integral $J_0=J+\lambda_{\alpha 0}J_\alpha$ along the arc $y_i(x, b)$ of the family (7). By the use of the Euler-Lagrange equations (3) and the properties of δy_i just established, it is found that along E_0 one has

$$dJ_0 = \int_{x_1}^{x_2}\{F_{y_i}\delta y_i + F_{y_i'}\delta y_i'\}dx = F_{y_i'}\delta y_i\Big|_{x_1}^{x_2} = 0, \qquad d^2J_0 = Q(\delta y) > 0$$

for all values $(db)\neq(0)$. In view of these relations and the equations $J_\alpha(b)=l_\alpha$ it follows that for values $(b)\neq(b_0)$ near $(b)=(b_0)$ the inequality

$$0 < J_0(b) - J_0(b_0) = J(b) - J(b_0)$$

holds, as was to be proved.

The proof of Lemma 2 will be complete if we show that the determinant $|\partial J_\alpha/\partial\lambda_\beta|$ is different from zero at $(b, \lambda)=(b_0, \lambda_0)$. To prove this let $\delta' Y_i(x) = Y_{i\lambda_\beta}(x, b_0, \lambda_0)d\lambda_\beta$. For the function $J_\alpha(b, \lambda)$ one then has along E_0

$$\delta' J_\alpha = \frac{\partial J_\alpha}{\partial\lambda_\beta} d\lambda_\beta = L_\alpha(\delta' Y), \tag{9}$$

where the functions L_α are given by equations (5). From the relations

$$Y_i(x_1, b, \lambda) = y_{i1}, \qquad Y_i(t_s, b, \lambda) = b_{is}, \qquad Y_i(x_2, b, \lambda) = y_{i2}$$

it follows by differentiation that

$$\delta' Y_i(x_1) = 0, \qquad \delta' Y_i(t_s) = 0, \qquad \delta' Y_i(x_2) = 0. \tag{10}$$

Moreover along E_0 we have

$$K_i(\delta' Y) + P_{i\beta} d\lambda_\beta = 0, \tag{11}$$

where $P_{i\beta}$, K_i are given by equations (4) and (6). This can be seen by substituting the functions (8) in the Euler-Lagrange equations (3), differentiating for λ_β, multiplying by $d\lambda_\beta$, and summing. Suppose now that the determinant in question were zero. Then there would exist constants $d\lambda_\beta$ not all zero such that $\delta' J_\alpha=0$ along E_0. If $\delta' Y_i\equiv 0$ on x_1x_2, one would have $P_{i\beta}d\lambda_\beta\equiv 0$ on x_1x_2 by equations (11), and E_0 could not be normal. Hence $\delta' Y_i\not\equiv 0$ on x_1x_2. By the use of equations (9), (10), (11), and $\delta' J_\alpha=0$ and integration by parts it would be found that along E_0 we would have $L_\alpha(\delta' Y)=0$ and

$$0 = \int_{x_1}^{x_2} \delta' Y_i[K_i(\delta' Y) + P_{i\beta} d\lambda_\beta]dx = Q(\delta' Y) + \delta' J_\beta d\lambda_\beta = Q(\delta' Y),$$

contrary to our assumption concerning the value of the second variation $Q(\eta)$ along E_0. Hence $|\partial J_\alpha/\partial\lambda_\alpha|\neq 0$ on E_0 and Lemma 2 is established.

We are now in position to prove Theorem 1. To do so we let $\mathfrak{J}'$ be a neighborhood of E_0 in xy-space so small that each subarc of the family (7) in $\mathfrak{J}'$ with end points on successive hyperplanes $x=t_{r-1}$, $x=t_r$ affords a minimum to the integral I_λ relative to admissible arcs in $\mathfrak{J}'$

joining its end points. This is possible by virtue of Lemma 1. Let $\mathcal{J}$ be a second neighborhood of E_0 interior to $\mathcal{J}'$ such that every admissible arc C in $\mathcal{J}$ joining the points 1 and 2 and satisfying equations (2) cuts the hyperplanes $x=t_s$ in points (t_s, b_{is}) whose y-coordinates b_{is} determine an extremal E_b of the family (7) lying in $\mathcal{J}'$. By Lemma 1 we have $I_\lambda(C) \geqq I_\lambda(E_b)$, the multipliers λ_α being those belonging to E_b. But since the arcs C and E_b satisfy equations (2), this implies that

$$I_\lambda(C) - I_\lambda(E) = J(C) - J(E_b) \geqq 0,$$

the equality holding only in case $C \equiv E_b$. Diminish $\mathcal{J}$ if necessary so that $J(E_b) \geqq J(E_0)$, as described in Lemma 2. We then have $J(C) \geqq J(E_b) \geqq J(E_0)$, the equality holding in both cases only in case $C \equiv E_0$. This proves the theorem.

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A NEW SUMMATION METHOD FOR DIVERGENT SERIES*

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1. **Introduction.** The method to be given here is a modification of that due to Euler-Knopp.† For the weighted means of the partial sums we use the binomial coefficients, but instead of beginning with the first we begin with the "central" one, that is with the greatest. Thus the initial terms always receive the greatest weight, as in the Cesàro-Hölder method.

In this paper it is shown (1) that this new method includes the first two Cesàro methods, and (2) that it also includes the first Euler-Knopp method; further, (3) the exact range of summability of the geometric series is determined. Finally, an example is given which indicates that this method may be more powerful than all those of Cesàro-Hölder, although this statement has not yet been proved.

2. **Definitions and notation.** Throughout we consider a series $\sum_{k=0}^{\infty} a_k$ and denote by S_n the sum of its first $n+1$ terms. We define σ_n as follows:

$$\sigma_n = \frac{1}{4^n} \sum_{k=0}^{n} C_{2n+1,n-k} S_k, \tag{1}$$

where $C_{n,k}$ denotes the ordinary binomial coefficient. If σ_n approaches

* Presented to the Society, April 11, 1936. See abstract 42-5-139.

† K. Knopp, Mathematische Zeitschrift, vol. 15 (1922), pp. 226–253.