# A PROPERTY OF HARMONIC FUNCTIONS IN THREE VARIABLES*

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A harmonic function $u(x, y)$ of the two variables $x$, $y$, which is defined in a circle, and which has a normal derivative vanishing on an open set of points on the circle, may be uniquely extended to all points exterior to the circle by assigning the same value to $u$ at any point $P'$ outside the circle as is assumed by $u$ at the point $P$, image of $P'$ with respect to the circle. The resulting function is harmonic in the entire plane except on the complement of the given open set with respect to the circumference of the circle. In this paper it will be investigated whether or not an analogous result holds for harmonic functions of three variables.

Let $r$, $\phi$, $\theta$ be spherical coordinates, and let $S$ be the sphere $r=1$. We consider a function $u(r, \phi, \theta)$ which is harmonic in the interior of the sphere and which together with its normal derivative is continuous on $S$. We suppose that $\Omega$ is a set of points of $S$, open with respect to $S$, on which $(\partial u/\partial r)=0$. If $M$ is a point interior to the sphere, then $M'$ is the image of $M$ with respect to $S$; that is, $M'$ is on the ray $OM$, where $O$ represents the origin, and $OM\cdot OM'=1$. On occasion the notations $u(r)$ and $u(r')$ will be used; they refer actually to $u(r, \phi, \theta)$ and $u(r', \phi, \theta)$, where $(r, \phi, \theta)$ and $(r', \phi, \theta)$ are images with respect to $S$, so that $rr'=1$. The variables $\phi$ and $\theta$ are omitted when there is no cause for confusion. The symbol $\Omega$ is used to denote either the set of points on the sphere or the set of values of $\phi$ and $\theta$ corresponding to these points; in each case the meaning will be clear. If a point $M(r, \phi, \theta)$ is under discussion, $Q$ will denote the point on $S$ on the ray $OM$.

Let $\Sigma$ be an open domain containing $S$ and its interior. We have the following theorem:

THEOREM 1. *A necessary and sufficient condition that there exist a unique analytic extension of $u$ across $\Omega$ into the portion of $\Sigma$ exterior to $S$ is that on $\Omega$*

$$\int_0^1 u(r)dr = \textit{constant}.$$

Suppose that $u$ is extensible across $\Omega$ as described. Let $v(r')$ be the

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function defined in the part of $\Sigma$ outside $S$ which is the harmonic extension of $u(r)$.

LEMMA 1. *On $S$, the function $w(Q)=v(Q)-u(Q)$ satisfies the equation*

$$\Delta_2 w(Q) = \left(\frac{\partial u}{\partial r}\right)_Q, \tag{1}$$

*where $\Delta_2$ is the second differential parameter of Beltrami,*

$$\Delta_2 = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

Consider the function $V(r, \phi, \theta)=r(\partial u/\partial r)$. It is defined and harmonic inside $S$; in fact,

$$\Delta^2\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}(r^2\Delta^2 u) = 0.$$

Furthermore $V$ vanishes on $\Omega$. Its harmonic extension across $\Omega$ is given by the formula

$$V(M') = -V(M)\cdot OM \qquad \text{or} \qquad V(r') = -rV(r).$$

Inside $S$, $V(r)=r(\partial u/\partial r)$, and by analytic extension $V(r)$ will be representable in this form everywhere. Outside $S$ then, $V(r')=r'[\partial v(r')/\partial r']$. Equating the two expressions obtained for $V(r')$, we have

$$r'\frac{\partial v(r')}{\partial r'} = V(r') = -rV(r) = -r^2\frac{\partial u(r)}{\partial r},$$

or

$$\frac{\partial v(r')}{\partial r'} = -r^3\frac{\partial u(r)}{\partial r}.$$

Now

$$v(r') = v(Q) + \int_1^{r'}\frac{\partial v}{\partial r'}dr' = v(Q) + \int_1^{r'} -r^3\frac{\partial u(r)}{\partial r}dr'$$
$$= v(Q) + \int_1^r r\frac{\partial u}{\partial r}dr.$$

On integrating this by parts, we obtain

$$v(r') = v(Q) - u(Q) + ru(r) - \int_1^r udr. \tag{2}$$

Since $v$ is harmonic, it satisfies Laplace's equation,

$$\Delta^2 v(r') = \frac{1}{r'^2}\left\{\frac{\partial}{\partial r'}\left(r'^2 \frac{\partial v}{\partial r'}\right) + \Delta_2 v\right\} = 0. \tag{3}$$

If we put $\partial v(r')/\partial r' = -r^3(\partial u(r)/\partial r)$, since $\partial/\partial r' = (\partial/\partial r)(dr/dr') = -r^2(\partial/\partial r)$, we see that (3) reduces to

$$r'^2\Delta^2 v(r') = r^2 \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \Delta_2 v = 0. \tag{3'}$$

From (2), we obtain $\Delta_2 v(r') = \Delta_2\{v(Q) - u(Q)\} + r\Delta_2 u(r) - \int_1^r \Delta_2 u dr$. Since $u$ is harmonic,

$$\Delta_2 u = -\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right).$$

Integrating $\Delta_2 u$ from 1 to $r$, we have

$$\int_1^r \Delta_2 u dr = -\int_1^r \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) dr = -r^2 \frac{\partial u}{\partial r} + \left(\frac{\partial u}{\partial r}\right)_Q.$$

By employing these values of $\Delta_2 u$ and $\int_1^r \Delta_2 u dr$ we reduce $\Delta_2 v(r')$ to the form

$$\begin{aligned}\Delta_2 v(r') &= \Delta_2\{v(Q) - u(Q)\} - r\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + r^2 \frac{\partial u}{\partial r} - \left(\frac{\partial u}{\partial r}\right)_Q \\ &= \Delta_2\{v(Q) - u(Q)\} - r^2 \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) - \left(\frac{\partial u}{\partial r}\right)_Q.\end{aligned}$$

If we insert this last value for $\Delta_2 v(r')$ into (3'), it results immediately that

$$r'^2\Delta^2 v(r') = \Delta_2\{v(Q) - u(Q)\} - \left(\frac{\partial u}{\partial r}\right)_Q = 0. \tag{4}$$

If we put $w(Q) = v(Q) - u(Q)$, equation (4) implies that $\Delta_2 w = (\partial u/\partial r)_Q$, and the proof of the lemma is completed.

Let $G(Q)$ and $H(Q)$ be any functions on $S$ with continuous second derivatives, and let $D$ be a domain on $S$ bounded by a curve $C$ with continuous normal. We have the relation*

$$\int_C \left(H \frac{\partial G}{\partial n} - G \frac{\partial H}{\partial n}\right) ds - \iint_D (H\Delta_2 G - G\Delta_2 H) dS = 0. \tag{5}$$

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* See for example Hadamard, *Leçons sur la Propagation des Ondes*, Paris, 1903, p. 49.

The differentiation $\partial/\partial n$ is with respect to the outer normal to $C$. Let $P$ be a point of $S$, and let $C$ be a small circle about $P$. Let $D$ be the larger of the two domains determined. For $H$ take the function $w(Q)$ as defined in Lemma 1. Then $\Delta_2 w(Q)$ is $(\partial u/\partial r)_Q$, and we denote this latter function by $F(Q)$. For $G(Q)$ we take the function $\log \sin \psi/2$ where $\psi = \sphericalangle(OP, OQ)$. By differentiation, $\Delta_2 \log \sin \psi/2$ is found to be $-1/2$, and the second integral of (5) reduces to

$$-\frac{1}{2}\iint_D w(Q)dS_Q - \iint_D F(Q) \log \sin \frac{\psi}{2}\, dS_Q \cdot$$

If the radius of $C$ is allowed to approach zero, it is shown without difficulty that the first integral of (5) approaches $-2\pi w(P)$. We then have

$$w(P) = \frac{1}{4\pi}\iint_S w(Q)dS_Q + \frac{1}{2\pi}\iint_S F(Q) \log \sin \frac{\psi}{2}\, dS_Q. \tag{6}$$

The first term of the right-hand member of (6) does not depend on $P$. On $\Omega$, we have $w(P)=0$, since there $v(P)=u(P)$. Consequently, we obtain

$$\iint_S F(Q) \log \sin \frac{\psi}{2}\, dS_Q = \text{constant} \tag{7}$$

for $P$ on $\Omega$.

LEMMA 2. *If a continuous function $F(Q)$ represents the values assumed by the normal derivative of a harmonic function $u$ on $S$, then*

$$\int_0^1 u(r)dr = -\frac{1}{2\pi}\iint_S F(Q) \log \sin \frac{\psi}{2}\, dS_Q + u(0). \tag{8}$$

Here the integration is performed along the ray $OP$, and as before, $\psi = \sphericalangle(OP, OQ)$.

Poisson's formula gives the value of a harmonic function $h(P)$ at a point $P$ interior to $S$ in terms of its values $h(Q)$ on $S$,

$$h(P) = \frac{1}{4\pi}\iint_S \frac{h(Q)(1-r^2)}{(1+r^2-2r\cos\psi)^{3/2}}\, dS_Q. \tag{9}$$

In particular, apply the formula (9) to the harmonic function $r(\partial u/\partial r)$, which assumes the values $(\partial u/\partial r)_Q = F(Q)$ on $S$. We have

$$r\frac{\partial u}{\partial r} = \frac{1}{4\pi}\iint_S \frac{F(Q)(1-r^2)}{(1+r^2-2r\cos\psi)^{3/2}}\, dS_Q,$$

or

$$\frac{\partial u}{\partial r} = \frac{1}{4\pi}\int\int_S \frac{F(Q)(1-r^2)}{r(1+r^2-2r\cos\psi)^{3/2}}\,dS_Q.$$

If $r$ and $\epsilon > 0$ represent two points inside $S$ on the same ray through $O$, then

$$\begin{aligned} u(r) - u(\epsilon) &= \frac{1}{4\pi}\int_\epsilon^r dt \int\int_S \frac{F(Q)(1-t^2)}{t(1+t^2-2t\cos\psi)^{3/2}}\,dS_Q \\ &= \frac{1}{4\pi}\int\int_S F(Q)dS_Q \int_\epsilon^r \frac{1-t^2}{t(1+t^2-2t\cos\psi)^{3/2}}\,dt. \end{aligned}$$

Since $u$ is harmonic, $\int\int_S F(Q)dS_Q = 0$, and the above is the same as $u(r) - u(\epsilon)$

$$= \frac{1}{4\pi}\int\int_S F(Q)dS_Q\left\{\int_\epsilon^r \left(\frac{1-t^2}{t(1+t^2-2t\cos\psi)^{3/2}} - \frac{1}{t}\right)dt\right\}.$$

Evaluating the inner integral, we find

$$u(r) - u(\epsilon) = \frac{1}{4\pi}\int\int_S F(Q)dS_Q\left[\frac{2}{T} + \log\frac{t+T-1}{t(t+T+1)}\right]_\epsilon^r,$$

where $T = (1+t^2-2t\cos\psi)^{1/2}$. Let $\epsilon$ approach zero. The bracketed expression in the integral, evaluated for $t=\epsilon$, approaches $2+\log(1-\cos\psi)-\log 2$, and if this value is inserted, we obtain

$$u(r) - u(0)$$

$$(10) \qquad = \frac{1}{4\pi}\int\int_S F(Q)dS_Q\left\{\frac{2}{R} + \log\frac{r+R-1}{r(r+R+1)} - \log(1-\cos\psi)\right\},$$

where $R = (1+r^2-2r\cos\psi)^{1/2}$. By combining the last two terms of the right-hand member of (10) we can reduce it to the simpler form

$$(10') \quad u(r) - u(0) = \frac{1}{4\pi}\int\int_S F(Q)dS_Q\left\{\frac{2}{R} - \log(1+R-r\cos\psi)\right\}.$$

We are now in a position to calculate $\int_0^1 u\,dr$. From (10),

$$\int_0^1 u(r)dr = \frac{1}{4\pi}\int\int_S F(Q)dS_Q \int_0^1 \left\{\frac{2}{R} + \log\frac{r+R-1}{r(r+R+1)} - \log(1-\cos\psi)\right\}dr + u(0).$$

The three definite integrals involved in the right-hand member have the following values:

$$\int_0^1 \frac{2dr}{(1+r^2-2r\cos\psi)^{1/2}} = 2\log\left(1+\sin\frac{\psi}{2}\right) - 2\log\sin\frac{\psi}{2},$$

$$\int_0^1 \log\frac{r+R-1}{r(r+R+1)}dr = 2\log\sin\frac{\psi}{2} - 2\log\left(1+\sin\frac{\psi}{2}\right)+1,$$

$$\int_0^1 \log(1-\cos\psi)dr = \log(1-\cos\psi) = 2\log\sin\frac{\psi}{2}+\log 2.$$

Substituting for the three definite integrals their values as just computed, we have

$$\int_0^1 u dr = -\frac{1}{2\pi}\int\int_S F(Q)\log\sin\frac{\psi}{2}\,dS_Q + u(0),$$

and the lemma is proved.

From (7) and (8) it follows that $\int_0^1 u dr$ is constant on $\Omega$, and the necessity of the condition of the theorem is proved.

To prove the sufficiency, suppose that $\int_0^1 u dr = c$ on $\Omega$. Then for $Q$ on $S$, we take for $w(Q) = v(Q) - u(Q)$ the function $c - \int_0^1 u dr$. The function $v(r')$ as given by (2) will actually represent the harmonic extension of $u$ across $\Omega$ into the entire exterior of the sphere. For (2) is then

$$\begin{aligned} v(r') &= w(Q) + ru(r) - \int_1^r u dr = c - \int_0^1 u dr + ru(r) - \int_1^r u dr \\ &= c + ru(r) - \int_0^r u dr. \end{aligned} \tag{11}$$

This function $v(r')$ has the following three properties:

(a) $v(Q) = c + u(Q) - \int_0^1 u dr = u(Q)$ on $\Omega$;

(b) $(\partial v/\partial r')_{r'=1} = [-r^2(r(\partial u/\partial r) + u - u)]_{r=1} = -(\partial u/\partial r) = 0$ on $\Omega$;

(c) from (4),

$$\begin{aligned} \Delta^2 v(r') &= \frac{1}{r'^2}\left\{\Delta_2 w(Q) - \left(\frac{\partial u}{\partial r}\right)_Q\right\} = \frac{1}{r'^2}\left\{-\int_0^1 \Delta_2 u dr - \left(\frac{\partial u}{\partial r}\right)_Q\right\} \\ &= \frac{1}{r'^2}\left\{\int_0^1 \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)dr - \left(\frac{\partial u}{\partial r}\right)_Q\right\} = 0 \end{aligned}$$

outside $S$.

Properties (a), (b), and (c) show clearly that $v(r')$ is the harmonic extension of $u$ across $\Omega$. This completes the proof of the theorem.

It is of interest to note that we can prescribe the values of $\int_0^1 u\,dr$ on the entire sphere and determine the harmonic function $u$ giving these values. More precisely, we prove the following theorem:

THEOREM 2. *Let $M(P)$ be a function with continuous second derivatives defined on $S$. There exists a unique function $u$ defined and harmonic inside $S$ and such that $\int_0^1 u\,dr = M(P)$; and $u$ may be expressed in terms of $M(P)$ by means of an integral formula.*

If a function $u$ does exist as described, then

$$u(0) = \frac{1}{4\pi}\iint_S u(r)dS_Q,$$

where $r$ is any radius not exceeding one and $dS_Q$ is an element of the area of the unit sphere. Thus

$$u(0) = \int_0^1 u(0)dr = \frac{1}{4\pi}\iint_S \left\{\int_0^1 u(r)dr\right\} dS_Q = \frac{1}{4\pi}\iint_S M(Q)dS_Q.$$

From (8) we see that $F(Q)$, the value of the normal derivative of $u$ on $S$, must satisfy the equation

$$M(P) = -\frac{1}{2\pi}\iint_S F(Q)\log\sin\frac{\psi}{2}\,dS_Q + \frac{1}{4\pi}\iint_S M(Q)dS_Q. \tag{12}$$

To determine $F(P)$ perform the operation $\Delta_2$ on both sides of the integral equation (12). The first term of the right-hand member gives a term $-F(P)$ because of the singular integrand $\log\sin\psi/2$,* and in addition a term coming from the differentiation under the integral sign:

$$\Delta_2 M(P) = -F(P) + \frac{1}{4\pi}\iint_S F(Q)dS_Q.$$

From (12), however, we obtain

$$\iint_S M(P)dS_P = -\frac{1}{2\pi}\iint_S F(Q)dS_Q \iint_S \log\sin\frac{\psi}{2}\,dS_P$$
$$+ \iint_S M(P)dS_P.$$

* The argument is precisely analogous to that used in proving Poisson's equation for the logarithmic potential; see O. D. Kellogg, *Foundations of Potential Theory*, Berlin, 1929, p. 156. For information about the differential parameter $\Delta_2$, see Darboux, *Leçons sur la Théorie des Surfaces*, Paris, 1914, book 7, chap. 1.

Now $\iint_S \log \sin \psi/2 \, dS_r \neq 0$, for the integrand is of constant sign; therefore $\iint_S F(Q) dS_Q = 0$. Hence

$$F(P) = -\Delta_2 M(P). \tag{13}$$

If (12) has a solution $F(Q)$, that solution must be given by (13). That this $F(Q)$ actually satisfies (12) follows from (6). Using the values of $F(Q)$ obtained from (13) in the formula (10), we determine a harmonic function $u$ whose normal derivative on $S$ is $F(Q)$ and for which

$$\int_0^1 u dr = M(P).$$

In particular we can choose $M(P) = c_1$ on some open set $\Omega_1$, and $M(P) = c_2$ on another open set $\Omega_2$, with $\Omega_1 \cdot \Omega_2 = 0$, and give to $M$ elsewhere any convenient values making $M$ have continuous second derivatives. The harmonic function $u$ determined inside of $S$ by the use of (13) and (10) has the properties:

$$\int_0^1 u dr = c_1 \text{ on } \Omega_1,$$

$$\int_0^1 u dr = c_2 \text{ on } \Omega_2,$$

$$\left(\frac{\partial u}{\partial r}\right) = 0 \text{ on } \Omega_1 \text{ and } \Omega_2.$$

This last statement follows from the fact that $(\partial u/\partial r)_P = -\Delta_2 M(P)$. From (11), $u$ may be extended harmonically across $\Omega_1$ or $\Omega_2$ according to the respective formulas

$$u_1(r') = c_1 + ru(r) - \int_0^r u dr,$$

and

$$u_2(r') = c_2 + ru(r) - \int_0^r u dr.$$

The two extensions $u_1$ and $u_2$ differ by $c_1 - c_2$. This function $u$, then, is an example of a harmonic function whose normal derivative vanishes on the open set $(\Omega_1 + \Omega_2)$, and which may not be extended harmonically across this set into the exterior of $S$ in a unique fashion. If $u$ is continued across $\Omega_1$ into the exterior of $S$ and back across $\Omega_2$ into the interior of $S$, it suffers a jump of $(c_1 - c_2)$.

The case $\partial u/\partial r = c$, $(c \neq 0)$, on an open set $\Omega$ can be treated in the same manner and yields the curious result that in this case $u$ cannot be extended harmonically across $\Omega$ into a domain $\Sigma$ containing the sphere $S$. For suppose that $u$ could be so extended. Then we would consider

$$V(r) = r\frac{\partial u}{\partial r} - c,$$

a harmonic function vanishing on $\Omega$. Its extension across $\Omega$ is

$$V(r') = -rV(r) = -r^2\frac{\partial u}{\partial r} + cr.$$

Inside the sphere, $V = r(\partial u/\partial r) - c$. By its analyticity $V$ will be expressible everywhere in this form; hence

$$V(r') = r'\frac{\partial v}{\partial r'} - c,$$

where $v$ is the extension of $u$ into the part of $\Sigma$ exterior to $S$. If we equate the two values obtained for $V(r')$ we obtain

$$-r^2\frac{\partial u}{\partial r} + cr = V(r') = r'\frac{\partial v}{\partial r'} - c,$$

or

$$\frac{\partial v}{\partial r'} = cr + cr^2 - r^3\frac{\partial u}{\partial r}\cdot$$

Now

$$v(r') = v(Q) + \int_1^{r'}\frac{\partial v}{\partial r'}dr' = v(Q) - c\log r - c(r-1) + \int_1^r r\frac{\partial u}{\partial r}dr.$$

Just as we obtained (4), we obtain in this case

$$\Delta^2 v(r') = \frac{1}{r'^2}\left\{\Delta_2[v(Q) - u(Q)] - \left(\frac{\partial u}{\partial r}\right)_Q + c\right\}.$$

The function $w(Q) = v(Q) - u(Q)$ must then satisfy, everywhere on $S$,

$$\Delta_2 w(Q) = \left(\frac{\partial u}{\partial r}\right)_Q - c.$$

This last equality is impossible, for

$$\iint_S \left\{ \left( \frac{\partial u}{\partial r} \right)_Q - c \right\} dS_Q = -4\pi c,$$

while from (5), for $G \equiv 1$ and $H = w$, we have

$$\iint_S \Delta_2 w(Q) dS_Q = 0.$$

THEOREM 3. *If* $(\partial u/\partial r) = c \neq 0$ *on the open set* $\Omega$ *of the sphere* $S$, *and if* $\Sigma$ *is a domain containing* $S$ *and its interior, it is in no case possible to extend* $u$ *harmonically across* $\Omega$ *into the portion of* $\Sigma$ *exterior to* $S$.

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# ON THE CLASS OF METRICS DEFINING A METRISABLE SPACE*

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Suppose we are given a metrisable† space $E$. Let $M$ be the class of all allowable metrics on $E$. Let $M_b$, $M_c$, $M_B$, and $M_C$ be, respectively, the classes of metrics in which the space is bounded, complete, totally bounded, and totally complete. The purpose of this note is to obtain systematically all possible theorems which state the equivalence of some topological property of $E$ (such as compactness, or separability) to the existence or non-existence of metrics having some of the above properties. An example is the well known theorem:

*In order that* $E$ *be compact it is necessary and sufficient that it be complete in every allowable metric.*

The problem may also be stated as follows: Using the four definitions as principles of classification and noting the inclusions $M_b \supset M_B \supset M_b M_C$ and $M_c \supset M_C \supset M_c M_B$, we may represent $M$ as the sum of seven disjoint sets: (1) $M - M_b - M_c$, (2) $M_b - M_B - M_b M_c$, (3) $M_c - M_C$

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† A topological space will be called *metrisable* if it is possible to define its continuity properties by means of a metric. Any metric which serves this purpose will be called *allowable*, and the space in conjunction with such a metric will be called a *metric space*. A metric space will be called *bounded* if there is a finite upper bound to the distance between any pair of its points. It will be called *complete* if every Cauchy sequence converges. It will be called *totally bounded* if it is, for every positive number $e$, the sum of a finite number of sets of diameter less than $e$. It will be called *totally complete* if every bounded set is compact. See C. Kuratowski, *Topologie* I, pp. 82, 87, 91, 196.