

ON CERTAIN GROUPS OF BIRATIONAL CONTACT TRANSFORMATIONS*

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1. **Introduction.** Let x_i and u_i , ($i = 1, 2, 3$), be the projective coordinates in plane xu of a point and line, respectively, and y_i and v_i the coordinates of a point and line in plane yv . The planes may be superimposed. Let s be a transformation in mixed coordinates

$$(1) \quad \rho y_i = \phi_i(x, u), \quad \sigma v_i = \psi_i(x, u), \quad i = 1, 2, 3,$$

where the ϕ_i and the ψ_i are polynomials homogeneous in the x_i of degrees a and c , respectively, and are also polynomials in the u_i of degrees b and d , respectively. This transformation is said to be birational if from (1) it is possible to obtain the inverse s^{-1} given by

$$(2) \quad \rho' x_i = \phi'_i(y, v), \quad \sigma' u_i = \psi'_i(y, v), \quad i = 1, 2, 3,$$

where the ϕ'_i and ψ'_i are polynomials homogeneous in the y_i of degrees a' and c' , and in the v_i of degrees b' and d' , respectively, and

$$\rho'' y_i = \phi_i(\phi', \psi'), \quad \sigma'' v_i = \psi_i(\phi', \psi').$$

This transformation is a line element transformation[†] if each of the equations

$$\sum u_i x_i = 0, \quad \sum v_i y_i = 0$$

implies the other. An element transformation is a birational contact transformation if, in addition to the above requirements, each of the two systems of equations

$$\sum x_i du_i = \sum u_i dx_i = 0$$

and

$$\sum y_i dv_i = \sum v_i dy_i = 0$$

implies the other.[‡]

Simple examples of birational contact transformations in the plane

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† A line element transformation is not a contact transformation unless it preserves unions.

‡ The representation of line elements by means of six coordinates x_i, u_i is due to Clebsch who also gave the necessary and sufficient conditions that (1) be a contact transformation. Clebsch-Lindemann-Benoist, *Leçons sur la Géométrie*, Paris, 1883, vol. 3, p. 463.

are furnished by enlarged Cremona point transformations, Cremona line transformations, polar reciprocation, and combinations of these. That birational contact transformations, other than these types, exist was demonstrated by Fano who gave some examples.* Groups of birational contact transformations were first investigated by Autonne, who determined the finite linear and finite quadratic groups of birational transformations in the plane.†

It is the purpose of this paper to construct and study an infinite mixed group of birational contact transformations in S_2 . It will then be shown that an analogous group exists in S_n .

2. The abelian group of birational contact transformations.

We begin with the consideration of a certain directrix equation

$$(3) \quad \sum (y_i/x_i)^{1/k} = 0,$$

k an integer, ($|k| > 1$), which, it will be shown, determines a birational contact transformation belonging to a group of such transformations. In §4 these transformations will be generalized to r -space, and in turn the directrix equations of the generalized transformations will be derived.

By virtue of (3) there corresponds to every point y_i in the yv -plane a triangular-symmetric curve in the xu -plane.‡ Likewise, to every point x_i in the xu -plane corresponds a triangular-symmetric curve in the yv -plane. To a generic curve in either plane corresponds the envelope of a family of triangular-symmetric curves in the other plane.

Differentiating (3), first under the assumption that the y_i are constants and second that the x_i are constants, we obtain

$$(4) \quad \sum_{i=1}^3 y_i^{1/k} x_i^{-(1+k)/k} dx_i = 0,$$

and

$$(5) \quad \sum x_i^{-1/k} y_i^{(1-k)/k} dy_i = 0.$$

From (3) and (4) it follows that

* G. Fano, *Trasformazione di contatto birazionali del piano*, Rendiconti delle Reale Accademia Nazionale dei Lincei, (6), vol. 8 (1928), p. 445.

† L. Autonne, (1) *Groupes d'ordre fini des substitutions linéaires de contact*, Journal de Mathématique, (4), vol. 3 (1887); (2) *Groupes d'ordre fini contenus dans le groupe quadratique crémonien*, ibid., vol. 4 (1888), pp. 177, 407; (3) Autonne's investigations regarding groups in S_3 and S_n which appear in Annales de l'Université de Lyon, Science et Médecine, vol. 16, 1905.

‡ Triangular-symmetric curves are those of the type $(x_1/a_1)^m + (x_2/a_2)^m + (x_3/a_3)^m = 0$. See, for instance, G. Loria, *Spezielle algebraische und transzendente ebene Kurven*, 1910, vol. 1, p. 341.

$$(6) \quad \begin{aligned} \rho y_1 &= x_1^{k+1}(x_2 dx_3 - x_3 dx_2)^k, \\ \rho y_2 &= x_2^{k+1}(x_3 dx_1 - x_1 dx_3)^k, \\ \rho y_3 &= x_3^{k+1}(x_1 dx_2 - x_2 dx_1)^k; \end{aligned}$$

and from (3) and (5) that

$$(7) \quad \begin{aligned} \rho' x_1 &= y_1^{-(k-1)}(y_2 dy_3 - y_3 dy_2)^{-k}, \\ \rho' x_2 &= y_2^{-(k-1)}(y_3 dy_1 - y_1 dy_3)^{-k}, \\ \rho' x_3 &= y_3^{-(k-1)}(y_1 dy_2 - y_2 dy_1)^{-k}. \end{aligned}$$

Since

$$\begin{aligned} u_1:u_2:u_3 &= (xdx)_1:(xdx)_2:(xdx)_3, \\ v_1:v_2:v_3 &= (ydy)_1:(ydy)_2:(ydy)_3, \end{aligned}$$

we find, by substituting in (6) and (7) and then solving for v_i and u_i , that the equations of the birational contact transformation T_k and of its inverse T_k^{-1} are

$$\begin{aligned} \rho y_i &= x_i^{k+1} u_i^k, & i &= 1, 2, 3, \\ T_k: \quad \sigma v_1 &= (x_2 x_3)^k (u_2 u_3)^{k-1}, & \sigma v_2 &= (x_3 x_1)^k (u_3 u_1)^{k-1}, \\ & \sigma v_3 &= (x_1 x_2)^k (u_1 u_2)^{k-1}, \end{aligned}$$

and

$$\begin{aligned} \rho' x_1 &= (y_2 y_3)^{k-1} (v_2 v_3)^k, & \rho' x_2 &= (y_3 y_1)^{k-1} (v_3 v_1)^k, \\ T_k^{-1}: \quad \rho' x_3 &= (y_1 y_2)^{k-1} (v_1 v_2)^k, \\ \sigma' u_i &= v_i^{k+1} y_i^k, & i &= 1, 2, 3. \end{aligned}$$

From the equations of T_k it can be easily verified that

$$\begin{aligned} \rho \sigma \sum v_i y_i &= (x_1 x_2 x_3)^k (u_1 u_2 u_3)^{k-1} \sum u_i x_i, \\ \rho \sigma \sum v_i dy_i &= (x_1 x_2 x_3)^k (u_1 u_2 u_3)^{k-1} [(k+1) \sum u_i dx_i + k \sum x_i du_i], \\ \rho \sigma \sum y_i dv_i &= -(x_1 x_2 x_3)^k (u_1 u_2 u_3)^{k-1} [k \sum u_i dx_i + (k-1) \sum x_i du_i]. \end{aligned}$$

Evidently T_k is a birational contact transformation when $k=0, 1$, so that the restriction previously put on k may be removed.

The triangular-symmetric curve

$$(8) \quad (a_1 x_1)^p + (a_2 x_2)^p + (a_3 x_3)^p = 0$$

is transformed by T_k into another triangular-symmetric curve, namely,

$$(8') \quad \sum (a_i y_i)^{p/(k+1)} = 0;$$

hence T_k leaves a three-parameter family of triangular-symmetric curves invariant. If $p' = p/(kp+1)$, the indices* of (8) and (8') are connected by the equation

$$\frac{1}{p'} - \frac{1}{p} = k.$$

Since T_0 transforms a generic element into itself, it is an identical transformation and will be represented by the symbol I . We note, furthermore, that

$$T_k^{-1} = T_{-k}, \quad T_k T_{-k} = I, \quad T_k T_m = T_m T_k = T_{k+m}.$$

Consequently, the transformations T_k , ($k = \dots, -2, -1, 0, 1, 2, \dots$), form a commutative group, G .

THEOREM 1. *The set of ∞^1 transformations T_k (k an integer) forms a discrete abelian group of birational contact transformations that leave a family of ∞^3 triangular-symmetric curves invariant.*

Let the vertices of the fundamental triangle be $A_1 (1, 0, 0)$, $A_2 (0, 1, 0)$, $A_3 (0, 0, 1)$, and let (x, u) be a generic line element. If A, B, C represent lines joining x to A_1, A_2, A_3 , respectively, and if D represents u , then the coordinates of these four lines are given by

$$\begin{aligned} A: & \quad 0, x_3, -x_2, & B: & \quad x_3, 0, -x_1, \\ C: & \quad x_2, -x_1, 0, & D: & \quad u_1, u_2, u_3. \end{aligned}$$

Evidently

$$x_3C = -x_1A + x_2B, \quad x_3D = u_2A + u_1B.$$

Therefore

$$R(AB, CD) = -\frac{x_2u_2}{x_1u_1}.$$

If (y, v) is the line element that corresponds to (x, u) by virtue of T_k , and if A', B', C' represent lines yA_1, yA_2, yA_3 , respectively, while D' represents v , then the coordinates of A', B', C', D' are as follows:

$$\begin{aligned} A': & \quad 0, x_3^{k+1}u_3^k, -x_2^{k+1}u_2^k, \\ B': & \quad x_3^{k+1}u_3^k, 0, -x_1^{k+1}u_1^k, \\ C': & \quad x_2^{k+1}u_2^k, -x_1^{k+1}u_1^k, 0, \\ D': & \quad (x_2x_3)^k(u_2u_3)^{k-1}, (x_3x_1)^k(u_3u_1)^{k-1}, (x_1x_2)^k(u_1u_2)^{k-1}. \end{aligned}$$

* G. Loria calls p the index of the curve (8); loc. cit., p. 329.

Therefore

$$\begin{aligned} x_3^{k+1}u_3^k C' &= -x_1^{k+1}u_1^k A' + x_2^{k+1}u_2^k B', \\ x_3 u_3 D' &= x_1^k u_1^{k-1} A' + x_2^k u_2^{k-1} B'. \end{aligned}$$

Consequently,

$$R(A'B', C'D') = -\frac{x_2 u_2}{x_1 u_1} = R(AB, CD).$$

In the same way it can be shown that the cross ratio that x forms with the points at which u intersects the sides of the fundamental triangle is equal to the cross ratio that the point y of the corresponding line element forms with the points at which v intersects the sides of the fundamental triangle.

THEOREM 2. *Every transformation in the group G sets up a one-to-one correspondence between the line elements of the plane in such manner that, if (x, u) and (y, v) are corresponding elements, (1) the cross ratio that u forms with the lines joining x to the vertices of the fundamental triangle equals the cross ratio that v forms with the lines joining y to the vertices of the fundamental triangle, and dually, (2) the cross ratio that x forms with the points at which u intersects the sides of the fundamental triangle equals the cross ratio that y forms with the points at which v intersects the sides of the fundamental triangle.*

The W curves

$$(9) \quad x_1^{a_1} x_2^{a_2} x_3^{a_3} = C, \quad a_1 + a_2 + a_3 = 0,$$

may be regarded as limits of the triangular-symmetric curves

$$a_1 x_1^m + a_2 x_2^m + a_3 x_3^m = 0, \quad a_1 + a_2 + a_3 = 0, \quad a_i \text{ real,}$$

when $m \rightarrow 0$.* Moreover, each of these W curves has the property that the cross ratio of any one of its points and the three points of intersection of the tangent at the point with the sides of the fundamental triangle is constant for the curve; and dually, the cross ratio of any one of its tangents and the three lines joining the point of tangency to the vertices of the fundamental triangle is constant for the curve.†

* Cesàro-Kowalewski, *Vorlesungen über natürliche Geometrie*, Leipzig, 1901, p. 130.

† The cross ratio associated with any W curve is independent of C , its value being $-a_i/a_j$, ($i \neq j$), where the subscripts depend on the sequence of the elements forming the cross ratio. To a given cross ratio correspond ∞^1 curves of the family (9), so that (9) constitutes a two-parameter family. It should be noted that only the ratios of the a_i and not the a_i themselves are significant as parameters. See F. Klein, *Vorlesungen über Höhere Geometrie*, Berlin, 1926, p. 170.

Consequently we can add the following corollary to Theorem 2:

COROLLARY. The birational contact transformations of group G transform each member of a two-parameter family of W curves into another member of the same family possessing the same characteristic cross ratio.

3. The ∞^1 involutory birational contact transformations. The directrix equation

$$\sum_{i=1}^3 (y_i/u_i)^{1/k} = 0$$

determines a birational contact transformation S_k the equations of which are obtained in the same manner as were the equations of T_k :

$$S_k: \begin{aligned} \rho y_i &= x_i^k u_i^{k+1}, \\ \sigma v_1 &= (x_2 x_3)^{k-1} (u_2 u_3)^k, & \sigma v_2 &= (x_3 x_1)^{k-1} (u_3 u_1)^k, \\ \sigma v_3 &= (x_1 x_2)^{k-1} (u_1 u_2)^k, & & i = 1, 2, 3, \end{aligned}$$

and

$$S_k^{-1}: \begin{aligned} \rho' x_i &= y_i^k v_i^{k+1}, \\ \sigma' u_1 &= (y_2 y_3)^{k-1} (v_2 v_3)^k, & \sigma' u_2 &= (y_3 y_1)^{k-1} (v_3 v_1)^k, \\ \sigma' u_3 &= (y_1 y_2)^{k-1} (v_1 v_2)^k, & & i = 1, 2, 3. \end{aligned}$$

From the equations of S_k it follows that

$$\rho \sigma \sum y_i v_i = (x_1 x_2 x_3)^{k-1} (u_1 u_2 u_3)^k \sum x_i u_i,$$

and

$$\begin{aligned} \rho \sigma \sum y_i dv_i &= - (x_1 x_2 x_3)^{k-1} (u_1 u_2 u_3)^k [(k-1) \sum u_i dx_i + k \sum x_i du_i], \\ \rho \sigma \sum v_i dy_i &= (x_1 x_2 x_3)^{k-1} (u_1 u_2 u_3)^k [k \sum u_i dx_i + (k+1) \sum x_i du_i]. \end{aligned}$$

It is apparent that S_k is an involutory birational contact transformation. By means of S_k the ∞^3 triangular-symmetric curves (8) are transformed into

$$(8'') \quad \sum_{i=1}^3 (y_i/a_i)^{p/(p+k+p-1)} = 0.$$

If the index of (8'') is represented by p' , it is seen that the indices of a triangular-symmetric curve and its transform by S_k are connected by the equation

$$\frac{1}{p} + \frac{1}{p'} = k + 1.$$

Four triangular-symmetric curves are left invariant by S_k , namely,

$$\sum_{i=1}^3 \pm (x_i)^{2/(k+1)} = 0.$$

Regard S_k as a substitution, and let $S_k S_n$ represent the effect of substitution S_k followed by S_n . Let P represent the enlarged polarity $\rho y_i = u_i$, $\sigma v_i = x_i$. Then it easy to show that

$$(10) \quad \begin{aligned} S_k S_n &= T_{k-n}, & S_k^2 &= I, & S_0 &= P, \\ S_n T_k &= S_{n-k}, & T_k S_n &= T_n S_k = S_{k+n}, \\ S_k T_k &= P, & S_k P &= T_k, & P S_k &= T_{-k}, \\ P S_k P &= S_{-k}, & S_n T_k S_n &= T_{-k}. \end{aligned}$$

The ∞^1 birational contact transformations S_k , (k an integer), do not form a group, but the aggregate set of the S_k and the T_n form a mixed group Γ of which G is evidently an invariant subgroup.

THEOREM 3. *The set of ∞^1 involutory birational contact transformations S_k and the set of ∞^1 birational contact transformations T_n together form a mixed group Γ which contains G as an invariant subgroup.*

For any integer c the set of transformations T_{kc} obviously forms a group G_1 , a subgroup of G . Moreover, the set of transformations in G_1 and the set S_{kc} together form a group Γ_1 , a subgroup of Γ . If we set up a one-to-one correspondence between the members of Γ_1 and Γ , such that T_{kc} and T_k correspond to one another and S_{kc} and S_k also correspond to one another for all integral values of k , it is seen, in view of the relationships given in (10), that Γ_1 and Γ are isomorphic and so are G_1 and G .

If (x, u) and (y, v) are a pair of corresponding line elements with respect to S_k , it can be shown, as it was for T_k , that the cross ratio formed by u and the lines joining x to the vertices of the fundamental triangle is equal to the cross ratio formed by v and the three lines joining y to the vertices of the fundamental triangle, and a similar result holds in the dual case.

THEOREM 4. *Every transformation in Γ preserves the cross ratio formed by the line of a generic line element and the joins of its point with the vertices of the fundamental triangle; and dually, every transformation in Γ preserves the cross ratio formed by the point of a generic line element and the intersections of its line with the sides of the fundamental triangle.*

It was shown above that the T_k leave the two-parameter family of

W curves (9) invariant. For the same reason the S_k leave the same family invariant.* Consequently, we add the following corollary:

COROLLARY. The transformations in Γ leave a two-parameter family of W curves invariant.

It is interesting to observe that S_n and T_n are determined by other directrix equations in addition to those indicated above.† For instance, the directrix equations

$$\sum (y_i x_i)^{1/(n+1)} = 0, \quad \sum (v_i u_i)^{1/(1-n)} = 0, \quad \sum (v_i/x_i)^{-1/n} = 0$$

determine S_n , whereas

$$\sum (y_i u_i)^{1/(n+1)} = 0, \quad \sum (v_i x_i)^{1/(1-n)} = 0, \quad \sum (v_i/u_i)^{-1/n} = 0$$

determine T_n .

4. Extension to higher space. The transformations T_n and S_n have analogs in higher spaces. Thus in r -space, the analog of T_n is given by

$$(11) \quad \bar{T}_n: \quad \rho y_i = x_i^{n+1} u_i^n, \quad \sigma v_i = \frac{1}{x_i^n u_i^{-1}}, \quad i = 1, 2, \dots, r + 1,$$

and the analog of S_n is given by

$$(12) \quad \bar{S}_n: \quad \rho y_i = x_i^n u_i^{n+1}, \quad \sigma v_i = \frac{1}{x_i^{n-1} u_i^n}, \quad i = 1, 2, \dots, r + 1.$$

From the first $r+1$ equations in (11) it follows that $\rho y_i/x_i = x_i^n u_i^n$. Therefore, using the relationship $\sum u_i x_i = 0$, we obtain a directrix equation of \bar{T}_n in the form

$$\sum_{i=1}^{r+1} (y_i/x_i)^{1/n} = 0.$$

From the second set of $r+1$ equations in (11) the directrix equation of \bar{T}_n in line coordinates is found to be

$$\sum_{i=1}^{r+1} (u_i/v_i)^{1/n} = 0.$$

Directrix equations of \bar{S}_n in mixed point and line coordinates, found

* A detailed proof appears below in connection with a discussion of these transformations in r -space.

† Autonne, *Journal de Mathématique*, vol. 4 (1888), shows that every contact transformation in mixed coordinates is determined by four directrix equations (primordiales), namely, point-punctual, point-linear, line-punctual, line-linear.

in the same manner, are

$$\sum_{i=1}^{r+1} (y_i/u_i)^{1/n} = 0$$

and

$$\sum_{i=1}^{r+1} (x_i/v_i)^{1/n} = 0.$$

The directrix equations of each of these transformations can be expressed in two other forms, as was indicated above in the planar case.

The aggregate of birational contact transformations, \bar{T}_n and \bar{S}_n (n an integer), can be shown to constitute a group $\bar{\Gamma}$ that has an invariant abelian subgroup \bar{G} composed of the transformations \bar{T}_n . The transformation \bar{S}_n is involutory, interchanges the ∞^{r+1} varieties in (13), replacing

$$(13) \quad \sum_{i=1}^{r+1} (a_i x_i)^{1/p} = 0$$

by

$$\sum (y_i/a_i)^{p/(pn+p-1)} = 0,$$

and leaves each variety in the set

$$\sum \pm (x_i)^{2/(n+1)} = 0$$

invariant. On the other hand, \bar{T}_n transforms the family (13) into itself, replacing (13) by

$$\sum (a_i y_i)^{p/(np+1)} = 0.$$

Because these statements can be proved in the same way that their analogues regarding S_n and T_n in the plane were proved, no verification is offered.

Let us consider the ∞^r varieties

$$(14) \quad F \equiv \prod_{i=1}^{r+1} y_i^{a_i} = K, \quad \sum_1^{r+1} a_i = 0,$$

K constant. This family is evidently a generalization in r -space of the W curves discussed above.* It will be proved that this family is left invariant by the transformations in $\bar{\Gamma}$. First we show that \bar{T} leaves (14) invariant.

* Surfaces of this type in three-space were investigated by Lie and Klein, Comptes Rendus de l'Académie des Sciences, 1870.

Let v_i , ($i=1, 2, \dots, r+1$), be the coordinates of the hyperplane tangent to F at y_i ; then

$$(15) \quad \alpha v_i = \frac{\partial F}{\partial y_i} = \frac{a_i K}{y_i},$$

where α is a constant of proportionality. Eliminating the v_i from (15) and from

$$\sigma v_i = \frac{1}{x_i^n u_i^{n-1}},$$

we obtain

$$\alpha' y_i = a_i x_i^n u_i^{n-1} K.$$

If we let the new constant of proportionality absorb the constant K , and if we drop the prime, this equation becomes

$$(16) \quad \alpha y_i = a_i x_i^n u_i^{n-1}.$$

The elimination of the u_i from (16) and $\rho y_i = x_i^{n+1} u_i^n$ results in

$$(17) \quad \rho' y_i = a_i^n x_i.$$

Substituting from (17) in (14) we obtain the transform of (14), namely

$$\prod_1^{r+1} a_i^{n a_i} \prod_1^{r+1} x_i^{a_i} = K, \quad \sum_1^{r+1} a_i = 0.$$

In the same way it can be proved that \bar{S}_n transforms (14) into

$$K \prod_1^{r+1} x_i^{a_i} = \prod_1^{r+1} a_i^{a_i(n+1)}, \quad \sum_1^{r+1} a_i = 0,$$

so that all the members of $\bar{\Gamma}$ transform this family into itself.*

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* Continuous (Lie) groups in which the discrete groups Γ and $\bar{\Gamma}$ are immersed will be considered in a subsequent paper.