ON THE PEANO CURVE OF LEBESGUE*

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Let $f(t)$ be the *even* continuous function of period two which is defined in the interval (0, 1) as follows: $f(t)=0$ in (0, 1/3), $f(t)=1$ in (2/3, 1), and $f(t)$ is linear in (1/3, 2/3). Our curve is defined by the parametric equations

$$(1)\qquad \begin{aligned} x(t) &= \frac{1}{2} f(t) + \frac{1}{2^2} f(3^2 t) + \frac{1}{2^3} f(3^4 t) + \cdots, \\ y(t) &= \frac{1}{2} f(3t) + \frac{1}{2^2} f(3^3 t) + \frac{1}{2^3} f(3^5 t) + \cdots, \quad 0 \leqq t \leqq 1. \end{aligned}$$

The inequalities $0 \leqq f(t) \leqq 1$ imply $0 \leqq x(t) \leqq 1$, $0 \leqq y(t) \leqq 1$, as well as the uniform convergence of both series (1), and hence imply the continuity of $x(t)$, $y(t)$. All there remains to show is that our curve will pass through an arbitrarily given point

$$(2)\ x_0 = \frac{a_0}{2} + \frac{a_2}{2^2} + \frac{a_4}{2^3} + \cdots, \quad y_0 = \frac{a_1}{2} + \frac{a_3}{2^2} + \frac{a_5}{2^3} + \cdots, \quad a_\nu = 0, 1,$$

of the square $0 \leqq x, y \leqq 1$, whose coordinates are given by their binary expansions. Indeed, let

$$(3)\qquad t_0 = \frac{2a_0}{3} + \frac{2a_1}{3^2} + \frac{2a_2}{3^3} + \cdots + \frac{2a_{k-1}}{3^k} + \frac{2a_k}{3^{k+1}} + \cdots.$$

If $a_0=0$, we have $0 \leqq t_0 \leqq 2/3^2+2/3^3+ \cdots =1/3$, hence $f(t_0)=0$; if $a_0=1$, we have $2/3 \leqq t_0 \leqq 2/3+1/3=1$, hence $f(t_0)=1$. In either case $f(t_0)=a_0$. Similarly $3^k t_0 =$ even integer $+ 2a_k/3 + 2a_{k+1}/3^2 + \cdots$ shows that

$$(4)\qquad f(3^k t_0) = a_k, \qquad k = 0, 1, 2, \cdots.$$

Now (1), (2), and (4) imply $x(t_0)=x_0$, $y(t_0)=y_0$.†

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† Lebesgue, *Leçons sur l'Intégration*, Paris, 1928, pp. 44–45, defines the functions $x(t)$, $y(t)$, first on Cantor's ternary set T, of points t_0 of the form (3), by means of the equations (2). Having proved their continuity on T, Lebesgue extends their definition throughout (0, 1) by linear interpolation over each one of the denumerable set of open intervals of which the set complementary to T is built up. It should be remarked that our curve (1) coincides with Lebesgue's curve within Cantor's set T but not on the complement of T.