If one weakens the requirements still further and only asks that a square be magic in the rows and columns, then a pair of antipodal elements can add up to 17 without the square being diabolic. This is illustrated by

1	15	4	14
12	2	9	11
8	7	16	3
13	10	5	6

which is magic in rows and columns, but not in diagonals, and which has $a+k=e+o=17$.

An analogous treatment of the problem of finding all diabolic magic squares is given by Kraitchik on page 167 of his book, *La Mathématique des Jeux*, where he shows that all diabolic magic squares can be derived by successive applications of A, B, C, and D from three particular ones which he gives.

CORNELL UNIVERSITY

A NOTE ON REGULAR BANACH SPACES*

B. J. PETTIS

Introduction. For an element x of a Banach space B_0† it is well known that the functional

$$X_x(f) = f(x)$$

defined over $B_1 = \overline{B}_0$, the Banach space composed of all linear functionals (real-valued additive and continuous functions) defined over B_0, is linear; moreover‡

$$\|X_x\|_{\overline{B}_1} = \|x\|_{B_0};$$

hence the additive operation $X_x = T(x)$ from B_0 to $B_2 = \overline{B}_1$ is continuous and norm-preserving. In B_2 let $B_2^{(0)}$ denote the set of image ele-

* Presented to the Society, October 30, 1937.

† S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 53. We shall use Banach's terminology.

‡ Banach, loc. cit., pp. 188–189.

ments of T. A space B_0 is *regular** if and only if

$$B_2 = B_2^{(0)}. \tag{0.1}$$

Examples of such spaces are L^p, l^p, $(1 < p < \infty)$, and the hyper-Hilbert spaces.

The purpose of the present note is to list in §1 some conditions necessary and sufficient for regularity, to give the form of the general linear operation from a regular space to the space l^1 of absolutely convergent series (§2), and to generalize some theorems already known concerning the differentiability of functions of bounded variation defined to a Banach space (§3). A function $Y(R)$, from the elementary figures in a euclidean figure R_0 to a B space, is of *bounded variation*, or BV, if and only if $\sum_{i=1}^{K}\| Y(R_i)\|$ is bounded over all finite sets of non-overlapping elementary figures $R_1, \cdots, R_K$ contained in R_0. The l.u.b. of such sums will be denoted by $\text{var}(Y; R_0)$. The last section includes the result (Theorem 7), analogous to that of J. A. Clarkson for uniformly convex spaces and those of Dunford and Morse,† that every function BV, from a linear interval to a regular B-space, is strongly differentiable‡ a.e. (almost everywhere).§

1. **Necessary and sufficient conditions for regularity.** We prove the following theorem:

THEOREM 1. *Each of the following conditions is necessary and sufficient that B_0 be regular:*

(1.1) *B_1 is regular;*

(1.2) *for linear sets in B_2 closure is equivalent to regular closure;*

(1.3) *B_0 is weakly complete, and for linear sets in B_2 regular closure is equivalent to weak closure as a set of functionals over B_1;*

(1.4) *$B_2^{(0)}$ is of the second category in B_2;*

(1.5) *each linear functional over $B_2^{(0)}$ is uniquely extensible over the whole of B_2.*

(0.1) implies (1.1). The operation $f_F = \overline{T}(F)$, adjoint (or associate)

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* H. Hahn, *Über lineare Gleichungssysteme in linearen Räumen*, Journal für die Reine und Angewandte Mathematik, vol. 157 (1927), pp. 214–229.

† J. A. Clarkson, *Uniformly convex spaces*, Transactions of this Society, vol. 40 (1936), pp. 396–414; N. Dunford and A. P. Morse, *Remarks on the preceding paper of James A. Clarkson*, ibid., pp. 415–420.

‡ The definition of strong and weak differentiability will be given in §3.

§ See also Gelfand, *Zur Theorie abstrakter Funktionen*, Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 17 (1937), Theorem 3. This theorem of Gelfand's was called to the writer's attention after the present paper had been accepted for publication.

to T and from $B_3 = \overline{B}_2$ to B_1, assigns to each $F(X)$ the functional $f_F(x)$ defined by

$$f_F(x) \equiv F(T(x)) \equiv F(X_x);$$

hence for every X_x in $B_2^{(0)}$

$$F(X_x) = f_F(x) = X_x(f_F).$$

Since $B_2^{(0)} = B_2$, this implies for every F in B_3 an f_F in B_1 such that $F(X) = X(f_F)$ for all X in B_2; thus B_1 is regular.

(1.1) implies (1.2). If M is a closed linear set in B_2, then for X_0 not in M there exists an F_0 in B_3 such that $F_0(X_0) = 1$, and $F_0(X) = 0$ for X in M. Since B_1 is regular, there is an f_0 in B_1 such that $F_0(X) = X(f_0)$ for all X in B_2. Hence $X_0(f_0) = 1$, $X(f_0) = 0$ for X in M, and M is a regularly closed set of functionals over B_1.

(1.2) implies (1.3). In general, regular closure implies weak closure as a set of functionals, and this, in turn, is stronger than ordinary closure. From (1.2) we then have closure, weak closure as a set of functionals, and regular closure, all equivalent for linear sets in B_2. Moreover, the closed linear subspace $B_2^{(0)}$ is therefore weakly closed as a set of functionals over B_1; hence B_0 is weakly complete.

(1.3) implies (0.1). From the weak completeness of B_0 it follows easily that $B_2^{(0)}$ is weakly closed as a set of functionals. From the remainder of (1.3) the linear set $B_2^{(0)}$ is regularly closed. Hence if there existed an X_0 in B_2 that was not in $B_2^{(0)}$, there would then be an f_0 such that $X_0(f_0) = 1$, while $X(f_0) = 0$ for all X in $B_2^{(0)}$. But then we would have $f_0(x) = 0$ for all x in B_0, that is, f_0 would be the identically zero functional over B_0, contrary to $X_0(f_0) = 1$.

That (1.4) is equivalent to (0.1) follows from the fact that the set of image points of a linear operation from one Banach space to another is either of the first category in the contradomain space of the operation or consists of the whole of that space.

If B_2 contains an X_0 not in $B_2^{(0)}$, then, since $B_2^{(0)}$ is linear and closed, there is an F_0 in B_3 such that $F_0(X_0) = 1$, and $F_0(X) = 0$ for X in $B_2^{(0)}$. Thus the zero functional over $B_2^{(0)}$ is extensible in at least two different ways, which contradicts (1.5).

THEOREM 2. *If B_0 is separable, then each of the following conditions is necessary and sufficient that B_0 be regular:*

(1.6)* *B_0 is weakly complete and weakly compact;*

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* (1.6) is well known; for the sufficiency see Banach, loc. cit., p. 189, and for the necessity, T. H. Hildebrandt, *Linear functional transformations in general spaces*, this Bulletin, vol. 37 (1931), p. 200.

(1.7) *B_0 is weakly complete and B_1 is separable;*

(1.8) *in B_1 weak convergence as a set of functionals is equivalent to weak convergence to an element as a set of elements.*

If B_0 is separable, so is $B_2^{(0)}$; hence if B_0 is in addition regular, then B_2, and therefore B_1, are separable. Thus from (1.3) the regularity and separability of B_0 together imply (1.7).

Condition (1.6) follows from (1.7). If $\{x_n\}$ is a bounded sequence in B_0, then, by diagonalizing, a subsequence $\{x_n'\}$ can be found such that $\lim_n f_i(x_n')$ exists for every f_i in a denumerable set dense in B_1. The bounded sequence $\{x_n'\}$ is then weakly convergent in B_0.

Thus if B_0 is separable, conditions (0.1), (1.6), and (1.7) are all equivalent.

The necessity of (1.8) is clear. If $\lim_n f_n(x)$ exists for every x, then $f_0(x)=\lim_n f_n(x)$ is a linear functional over B_0, and if B_0 is regular, then $F(f_n)$ converges to $F(f_0)$ for every F in B_2. If (1.8) holds and $F(f)$ is linear over B_1, then the convergence of $\{f_n\}$ to f_0, as a sequence of functionals over B_0, implies $\lim F(f_n)=F(f_0)$; hence $F(f)$ is weakly continuous over B_1. Since B_0 is separable, this insures the inclusion of F in $B_2^{(0)}$.†

THEOREM 3. *If B_0 is regular, so is every closed linear subspace B_0'.*

Suppose $X'(f')$ is a linear functional over the space B_1' conjugate to B_0'. Define $X(f)=X'(f')$, where f varies over B_1, and f' is the linear functional over B_0' determined by f. Since B_0 is regular, there exists an x in B_0 such that $X(f)=f(x)$ for every f in B_1. We have only to show that x is in B_0'; for then

$$X'(f') = X(f) = f(x) = f'(x)$$

for all f and hence for all f', since any f' can be extended to form an f. If x is not in the closed linear subset B_0' of B_0, there exists an f^* in B_1 such that $f^*(x)\neq 0$, $f^*(x')=0$ for all x' in B_0'. But then $X(f^*)=f^*(x)\neq 0$, while $X(f^*)=X'(f^{*\prime})=X'(0)=0$, a contradiction that completes the proof.

If B_0 is regular, then any separable closed linear subspace is also regular and is therefore weakly complete and weakly compact. The space B_0 must then have these two properties also. This gives us the following corollary:

COROLLARY. *A regular space is weakly complete and weakly compact.*

2. Operations from a regular space to the space l^1. We prove the following theorem:

† Banach, loc. cit., p. 131.

THEOREM 4. *If B_0 is regular and the operation*

$$T(f) = \{f^i\}$$

from B_1 to l^1 (space of absolutely convergent series) is linear, then T is completely continuous and is defined by a series $\sum_1^\infty x_n$, unconditionally convergent† in B_0; conversely, such a series defines a linear completely continuous operation from B_1 to l^1. The norm of T is $\text{l.u.b.}_{\|f\|=1}\sum_1^\infty |f(x_n)|$.

Given a linear T, let $F=\overline{T}(m)$ be the adjoint linear operation from $\overline{l^1}=l^\infty$, the space of bounded sequences, to B_2. If $F_i=\overline{T}(m_i)$, where m_i is the linear functional over l^1 defined by the sequence having 1 in the ith place and 0's elsewhere, then

$$F_i(f) = m_i(T(f)) = m_i(\{f^j\}) = f^i.$$

Since B_0 is regular, there exists an x_i such that

$$f^i = F_i(f) = f(x_i)$$

for all f in B_1. Hence

$$T(f) = \{f(x_i)\}. \tag{2.1}$$

From the weak completeness of B_0 and the fact that $\sum_{i=1}^\infty |f(x_i)| < \infty$ for all f, the series $\sum x_i$ is unconditionally convergent;‡ moreover, given $\epsilon>0$ there is an N_ϵ such that§

$$\sum_{N_\epsilon+1}^{\infty} | f(x_i) | < \epsilon \tag{2.2}$$

for all f with $\|f\| \leqq 2$. If $\|f_i^*\| \leqq 1$, then by diagonalizing we can find a subsequence $\{f_i\}$ such that $\lim_i f_i(x_n)$ exists for every n. This fact, combined with (2.2), gives

$$\sum_{n=1}^{\infty} | f_i(x_n) - f_j(x_n) | < 2\epsilon$$

for i, j greater than a suitably chosen K_ϵ. Thus $T(f)$ is completely continuous.

Conversely, suppose $\sum x_n$ converges unconditionally in B_0. The operation $T(f)$ in (2.1) is then defined and additive from B_1 to l^1. Since

† A series is unconditionally convergent if every reordering is convergent; W. Orlicz, *Beiträge zur Theorie der Orthogonalentwicklungen*, II, Studia Mathematica, vol. 1 (1929), pp. 241–255; p. 242. For a statement of equivalent definitions see Banach, loc. cit., p. 240.

‡ Orlicz, loc. cit., Theorem 2.

§ Orlicz, loc. cit., p. 246.

it is the "pointwise" limit of the linear operations

$$(2.3)\qquad T_n(f) = \{f(x_1), \cdots, f(x_n), 0, 0, \cdots\}, \qquad n = 1, 2, \cdots,$$

T must also be linear. The complete continuity follows from the first part of the proof.

The statement concerning the norm of T is an immediate consequence of (2.1).

Without resorting to (2.2) the complete continuity of T could have been proved by using the following facts: (i) T is linear, (ii) B_1 is weakly complete and weakly compact, (iii) the property of converging weakly to an element is preserved under linear operations, (iv) in l^1 weak and strong convergence are equivalent.

In view of (1.1), Theorem 4 can be stated as follows:

THEOREM 4'. *If B_0 is regular, an operation*

$$T(x) = \{f_x^i\}$$

from B_0 to l^1 is linear if and only if it is defined by an unconditionally convergent series $\sum f_i$ in B_1. If it is linear, it is completely continuous and

$$\| T \| = \underset{\|x\|=1}{\text{l.u.b.}} \sum_1^\infty | f_i(x) |.$$

It is clear from the forms of their general linear functionals that the spaces L^p, l^p, $(1 < p < \infty)$, are regular. Applying Theorem 4' and (2.2) to these spaces we obtain the following corollary:

COROLLARY.† *An operation $T(x)$ from L^p (l^p), $(1 < p < \infty)$, to l^1 is linear if and only if there exists in $L^{p'}$ $(l^{p'})$, $(p' = p/p-1)$, a sequence $\{y_i\} = \{y_i(s)\}$ $(\{y_i\} = \{\{y_i^j\}\})$ such that*

$$(1)\qquad \sum_{i=1}^\infty \left| \int_0^1 y_i(s)x(s)ds \right| < \infty \quad \left(\sum_{i=1}^\infty \left| \sum_{j=1}^\infty y_i^j x^j \right| < \infty \right)$$

for every $x = x(s)$ $(x = \{x^j\})$ in L^p (l^p); that is, $\sum_1^\infty y_i$ converges unconditionally in $L^{p'}$ $(l^{p'})$, and

$$(2)\qquad T(x) = \left\{ \int_0^1 y_i(s)x(s)ds \right\} \quad \left(T(x) = \left\{ \sum_j y_i^j x^j \right\} \right).$$

If T is linear, then

† In this corollary the case l^p to l^1 is already known. See H. R. Pitt, *A note on bilinear forms*, Journal of the London Mathematical Society, vol. 11 (1936), pp. 174–180; and L. W. Cohen and N. Dunford, *Transformations on sequence spaces*, Duke Mathematical Journal, vol. 3 (1937), pp. 689–701.

(i) *T is completely continuous;*
(ii) *given $\epsilon > 0$ there exists an N_ϵ such that*

$$\sum_{i=N_\epsilon}^{\infty} \left| \int_0^1 y_i(s)x(s)ds \right| < \epsilon \qquad \left(\sum_{i=N_\epsilon}^{\infty} \left| \sum_{j=1}^{\infty} y_i^j x^j \right| < \epsilon \right)$$

uniformly for all x with

$$\int_0^1 | x(s) |^p ds \leqq 1 \quad \left(\sum_{j=1}^{\infty} | x^j |^p \leqq 1 \right);$$

(iii)
$$\|T\| = \underset{\int_0^1 |x(s)|^p ds \leqq 1}{\text{l.u.b.}} \sum_{i=1}^{\infty} \left| \int_0^1 y_i(s)x(s)ds \right|$$

$$\left(\|T\| = \underset{\sum_{j=1}^{\infty} |x^j|^p \leqq 1}{\text{l.u.b.}} \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} y_i^j x^j \right| \right).$$

3. **Differentiation in regular spaces.** In this section we consider the differentiability of functions BV from a linear interval to a regular space. Our main purpose is to prove the theorem stated in the introduction. Every such function is strongly differentiable a.e. A function $Y(R)$, from the elementary figures in a euclidean figure R_0 to a Banach space B_0, is said to be *strongly differentiable* at a point s in the interior of R_0 if there exists an element $y(s)$ of B_0 such that for cubes I lying in R_0 and containing s it is true that $\| Y(I)/|I| - y(s)\|$ tends to 0 with $|I|$, the measure of I. The function $Y(R)$ is *weakly differentiable* at s if there exists a $y(s)$ such that

$$\lim_{\substack{|I|\to 0 \\ s \in I}} | f(Y(I)/I) - f(y(s)) | = 0$$

for every linear functional f.

THEOREM 5. *If $Y(R)$ is defined and BV from the figures in a euclidean figure R_0 to a Banach space B_0, and if $Y(R)$ is weakly differentiable a.e. in R_0 to the function $y(s)$, then $y(s)$ is Bochner integrable.**

Let I_0 be a cube containing R_0, and $\{\Pi_n\}$ a sequence of partitions of I_0 into a finite number of non-overlapping (except on their boundaries) and non-degenerate subcubes, with Π_{n+1} a repartition of Π_n and $\lim_{n\to\infty}$ norm $\Pi_n = 0$. If s lies in the interior of a cube I_{in} of Π_n and R_0 contains I_{in}, then define $y_n(s)$ as $Y(I_{in})/|I_{in}|$; otherwise let $y_n(s)$ vanish. Then a.e. in R_0 we have $\{y_n(s)\}$ converging weakly to $y(s)$, hence

* S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorräumes sind,* Fundamenta Mathematicae, vol. 20 (1933), pp. 262–276.

$$\|y(s)\| \leqq \liminf_n \|y_n(s)\|$$

a.e. in R_0. From Fatou's lemma and the inequality

$$\int_{R_0} \|y_n(s)\| ds \leqq \text{var}\ (Y;\ R_0) < \infty,$$

the real-valued $\liminf_n \|y_n(s)\|$ must be summable. The abstract function $y(s)$ is measurable in the sense of Bochner,* since a.e. in R_0 it is a weak derivative.† Hence $\|y(s)\|$ is measurable and summable, which implies that the measurable $y(s)$ is Bochner integrable.

COROLLARY.‡ *If, in addition to the assumptions in Theorem* 5, $Y(R)$ *is additive and AC (absolutely continuous), then*

$$Y(R) = \int_R y(s) ds, \tag{A}$$

and $Y(R)$ *is strongly differentiable a.e. to the function* $y(s)$.

In view of the strong differentiability a.e. of the Bochner integral,§ it is sufficient to prove (A). Let $Z(R) = \int_R y(s) ds$; then‖ $f(Z(R)) = \int_R f(y(s)) ds$ for every f in $B_1 = \overline{B}_0$ and every R in R_0. But $f(Y(R))$ is a real-valued additive and AC function that is differentiable a.e. to $f(y(s))$, and hence $f(Y(R)) = \int_R (d/ds f(Y(R))) ds = \int_R f(y(s)) ds$ for every f and R. Thus $Y(R) = Z(R)$ for every R.

Because of this corollary a theorem of Dunford and Morse¶ may be altered to read:

THEOREM 6. *A Banach space* B_0 *has the property* (DBV), *namely, that every BV function from a linear interval to* B_0 *is strongly differentiable a.e., and its derivative is Bochner integrable, if and only if every Lipschitzian function from a linear interval to* B_0 *is weakly differentiable a.e.*

THEOREM 7. *If* B_0 *is regular, then* B_0 *has property* (DBV).

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* Bochner, loc. cit., p. 263.

† See *On integration in vector spaces*, a paper by the present author which is to appear in the Transactions of this Society.

‡ Theorem 5 and the corollary are generalizations of theorems due to Clarkson, loc. cit., pp. 409–410.

§ Bochner, loc. cit., p. 269.

‖ N. Dunford, *Integration and linear operations*, Transactions of this Society, vol. 40 (1936), pp. 474–494; p. 475, Theorem 2.3.

¶ Loc. cit., p. 415.

Suppose $Y(t)$ is defined and Lipschitzian from a linear interval $[a, b]$ to a regular space B_0. From the Lipschitzian property it is clear that every value of $Y(t)$ lies in the separable closed linear subspace B_0' generated by the values of $Y(t)$ at the rationals. By (1.7) and Theorem 3 there exists a denumerable set $\{f_i'\}$ dense in the space B_1' adjoint to B_0'; and B_0' is weakly complete. Let $Y(R)$ be the additive figure function determined by $Y(t)$. Then $f_i'(Y(R))$ is a real-valued additive Lipschitzian figure function; hence there exists in $[a, b]$ a measurable set S with $|S| = b-a$ and such that s in S implies $f_i'(Y(R))$ differentiable at s for each i. Thus if s is in S, and if $\{I_n\}$ are intervals closing down on s, it follows that

$$\lim_n \frac{f'(Y(I_n))}{|I_n|} = \lim_n f'\left(\frac{Y(I_n)}{|I_n|}\right)$$

exists for every f' in a set dense in B_1'; moveover, by the Lipschitzian property $\|Y(I_n)/|I_n|\| \leqq K$. These two conditions imply that $\lim f'(Y(I_n)/|I_n|)$ exists for every f' in B_1', that is, $\{Y(I_n)/|I_n|\}$ is a weakly convergent sequence. Since B_0' is weakly complete there is an element $y(s)$ to which this sequence converges weakly. If $\{\bar{I}_n\}$ is another sequence of intervals closing down on s, then $\{Y(\bar{I}_n)/|\bar{I}_n|\}$ is also weakly convergent, and for every f'

$$\lim_n f'(Y(\bar{I}_n)/|\bar{I}_n|) = \lim_n f'(Y(I_n)/|I_n|) = f'(y(s)).$$

Thus $Y(R)$, as a function defined to B_0', is weakly differentiable a.e. in $[a, b]$. Since any f in B_1 defines an f' in B_1', $Y(R)$, as defined to B_0, has the same property. On application of Theorem 6, the proof is complete.

If L denotes the space of functions Lebesgue integrable on $[0, 1]$, we have the following corollary:

COROLLARY. *If B_0 is regular, then $T(\phi)$ from L to B_0 is linear if and only if there exists an essentially bounded and Bochner measurable function $x(s)$ on $[0, 1]$ to B_0 such that*

$$T(\phi) = \int_0^1 x(s)\phi(s)ds,$$

the integration being in the Bochner sense. The norm of T is ess.$_s$ sup. $\|x(s)\|$.

To obtain this from the theorem the reader is referred to a proof by Dunford.*

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* Loc. cit., p. 483, Theorem 5.3.