

SOME PROPERTIES OF FUNCTIONS OF EXPONENTIAL TYPE

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Suppose that $f(z)$ is an entire function such that

$$f(z) = O(e^{\lambda|z|}),$$

and on the real axis $f(z)$ is real and bounded by 1. First it is shown that the function $\cos \lambda z - f(z)$ cannot have complex zeros. Moreover its real zeros are simple at the points where the strict inequality $|f(z)| < 1$ is satisfied. This theorem is then used to find a "best possible" dominant over the complex plane of the class of functions $f(z)$. Finally it is shown that these results contain two theorems of S. Bernstein.

THEOREM 1. *Let $f(z)$ be an entire function of $z = x + iy$, real for real z , and satisfying the conditions:*

$$(1) \quad |f(x)| \leq 1$$

on the real axis, and

$$(2) \quad |f(z)| = O(e^{\lambda|z|}), \quad \lambda > 0,$$

uniformly over the entire plane. Then for every real α the function

$$\cos(\lambda z + \alpha) - f(z)$$

has only real zeros, or vanishes identically. Moreover all the zeros are simple, except perhaps at points on the real axis where $f(x) = \pm 1$.

In the proof of Theorem 1 we shall use the following result of Pólya and Szegő,* which we state as a lemma.

LEMMA 1. *If $f(z)$ satisfies the conditions of Theorem 1, then actually*

$$(3) \quad |f(z)| \leq e^{\lambda|y|}.$$

PROOF. By the hypotheses of Theorem 1 the function $f(z)e^{i\lambda z}$ is bounded on the positive halves of the real and imaginary axes and is $O(e^{\lambda|z|})$ in the angular region between them. Then by the Phragmén-Lindelöf† principle $f(z)e^{i\lambda z}$ is bounded throughout the first quadrant. In the same way one shows that it is bounded in the second quadrant.

* G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 36, prove this by a different method.

† E. C. Titchmarsh, *Theory of Functions*, p. 177.

Then $f(z)e^{\lambda z}$ is bounded by 1 on the real axis and by some constant in the upper half-plane. Again applying the Phragmén-Lindelöf principle, this time with the angular region the upper half-plane, we see that $|f(z)e^{\lambda z}| \leq 1$ in the upper half-plane, so (3) is true for $y \geq 0$. When we use the function $f(z)e^{-\lambda z}$ the same method shows that (3) is true for $y \leq 0$.

To prove Theorem 1 it will be sufficient to consider the function $\cos \lambda z - f(z)$, that is, the case $\alpha = 0$. Let ϵ be some positive number less than 1, and consider the function

$$f_\epsilon(z) = \frac{\sin \lambda \epsilon z}{\lambda \epsilon z} (1 - \epsilon)f((1 - \epsilon)z).$$

By Lemma 1,

$$(4) \quad |f_\epsilon(z)| \leq \frac{e^{\lambda \epsilon |y|}}{\lambda \epsilon |z|} (1 - \epsilon)e^{\lambda |y|(1-\epsilon)} < \frac{e^{\lambda |y|}}{\lambda \epsilon |z|};$$

so if y_0 is sufficiently large, we have, on the lines $y = \pm y_0$,

$$|f_\epsilon(z)| < |\cos \lambda z|.$$

It follows from (4) that, if K is a sufficiently large positive integer,

$$|f_\epsilon(z)| < |\cos \lambda z|$$

on the lines $x = \pm K\pi/\lambda$.

Let ζ be a closed rectangular contour consisting of segments of the lines $x = \pm K\pi/\lambda$, $y = \pm y_0$. We have shown that $|\cos \lambda z| > |f_\epsilon(z)|$ on ζ , so by Rouché's theorem* the function

$$\cos \lambda z - f_\epsilon(z)$$

has the same number of zeros in ζ as $\cos \lambda z$, that is, $2K$ zeros. On the real axis $|f_\epsilon(x)| \leq 1 - \epsilon$, so at the points $\nu\pi/\lambda$, ($\nu = 0, \pm 1, \pm 2, \dots$), we have $|f_\epsilon(z)| < |\cos \lambda z|$. Thus $\cos \lambda z - f_\epsilon(z)$ is alternately plus and minus at the $2K + 1$ points $\nu\pi/\lambda$, ($\nu = -K, -K + 1, \dots, K$); so inside ζ it has at least $2K$ real zeros. But we have shown that there are exactly $2K$ zeros of $\cos \lambda z - f_\epsilon(z)$ in ζ . Hence there are no complex zeros, and there is exactly one (simple) zero in each interval $(\nu\pi/\lambda, (\nu + 1)\pi/\lambda)$, ($\nu = -K, \dots, K - 1$). Taking larger values of y_0 and K we see that $\cos \lambda z - f_\epsilon(z)$ has exclusively real and simple zeros, which lie in the intervals $\nu\pi/\lambda < z < (\nu + 1)\pi/\lambda$, (ν integer, $-\infty < \nu < \infty$).

When $\epsilon \rightarrow 0$ the function $\cos \lambda z - f_\epsilon(z)$ approaches $\cos \lambda z - f(z)$

* E. C. Titchmarsh, loc. cit., p. 116.

uniformly in every bounded domain. But if the latter function is not identically zero it follows from a theorem of Hurwitz* that its zeros are limit points of the zeros of $\cos \lambda z - f_\epsilon(z)$. Thus $\cos \lambda z - f(z)$ cannot have non-real zeros; moreover it has an infinite number of real zeros which are all simple, except those at the points $\nu\pi/\lambda$ if $f(\nu\pi/\lambda) = (-1)^\nu$. Every interval $\nu\pi/\lambda < z < (\nu+1)\pi/\lambda$ at the endpoints of which $|f(z)| < 1$ contains exactly one zero. If $f(\nu\pi/\lambda) = (-1)^\nu$, we have a double zero at $\nu\pi/\lambda$ but no further zeros in the interior or at the endpoints of the interval $((\nu-1)\pi/\lambda, (\nu+1)\pi/\lambda)$. This proves Theorem 1.

We have shown that, if $f(z)$ satisfies the conditions of Theorem 1, then actually the inequality (3) is satisfied. There is, however, no such function for which (3) becomes an equality at points off the real axis. Using Theorem 1 we can show that the stronger inequality

$$(5) \quad |f(z)| \leq \cosh \lambda y,$$

is satisfied. If $f(z) \equiv \cos(\lambda z + \alpha)$, α real, then the equality holds along certain lines parallel to the imaginary axis.

THEOREM 2. *If $f(z)$ satisfies the conditions of Theorem 1, then*

$$(6) \quad |f(z)| \leq \cosh \lambda y$$

and, unless $f(z)$ is of the form $\cos(\lambda z + \alpha)$, the equality can occur only on the real axis.

PROOF. It will be sufficient to show that (6) is true on the imaginary axis. Suppose $f(z)$ is not of the form $\cos(\lambda z + \alpha)$, and for some y , ($|y| > 0$), we have

$$|f(iy)| \geq \cosh \lambda y.$$

From the expansion

$$\cos(\lambda iy + \beta) = \cos \beta \cosh \lambda y - i \sin \beta \sinh \lambda y$$

we see that $\cos(\lambda iy + \beta)$ has, for a suitable real β , the same amplitude as $f(iy)$. Since $|\cos(\lambda iy + \beta)| \leq \cosh \lambda y \leq |f(iy)|$, there is a real γ , ($0 < \gamma \leq 1$), such that $|\gamma f(iy)| = |\cos(\lambda iy + \beta)|$. Then

$$\cos(\lambda z + \beta) = \gamma f(z)$$

at the point $z = iy$, since the amplitudes and magnitudes are the same. But $\gamma f(z)$ satisfies the conditions of Theorem 1, so $\cos(\lambda z + \beta) - \gamma f(z)$ can have only real zeros. The contradiction proves Theorem 2.

* E. C. Titchmarsh, loc. cit., p. 119.

It has been shown by S. Bernstein* that if $f(z)$ satisfies the conditions of Theorem 1, then

$$|f'(z)| \leq \lambda$$

on the real axis. Szegö (for the case in which $f(z)$ is a trigonometric polynomial) and later Boas (under essentially the same conditions as in Theorem 1) obtained† the stronger inequality

$$(7) \quad \{f'(z)\}^2 + \lambda^2 \{f(z)\}^2 \leq \lambda^2$$

on the real axis. The authors obtained a generalization of (7) for complex values of z in a previous paper. Using Theorem 1 we can now prove this corollary:

COROLLARY. *If $f(z)$ satisfies the conditions of Theorem 1, then (7) is true on the real axis.*

PROOF. Suppose $f(z)$ satisfies the conditions of Theorem 1 and is not of the form $\cos(\lambda z + \alpha)$. At points of the real axis where $f(z) = \pm 1$ we must have $f'(z) = 0$, so (7) is certainly true. Hence suppose that, at some point z_0 , $|f(z_0)| < 1$, and (7) is not satisfied. Then by suitable choice of real γ , ($0 < \gamma \leq 1$), we have the equality

$$(8) \quad \{\gamma f'(z)\}^2 + \lambda^2 \{\gamma f(z)\}^2 = \lambda^2.$$

Then since $\cos(\lambda z + \alpha)$ satisfies the differential equation

$$\left[\frac{d}{dz} \cos(\lambda z + \alpha) \right]^2 + \lambda^2 [\cos(\lambda z + \alpha)]^2 = \lambda^2,$$

there is a real α so that, at the point z_0 ,

$$\cos(\lambda z + \alpha) = \gamma f(z), \quad \frac{d}{dz} \cos(\lambda z + \alpha) = \gamma f'(z).$$

Thus the function

$$\cos(\lambda z + \alpha) - \gamma f(z)$$

has a double zero at the point z_0 , where $|\gamma f(z_0)| < 1$; but by Theorem 1 this is impossible. The contradiction proves the corollary.

Using Theorem 2 we prove a second theorem‡ of S. Bernstein.

* S. Bernstein, *Comptes Rendus*, vol. 176 (1923), p. 1603.

† G. Szegö, *Schriften der Königsberger Gelehrten Gesellschaft, Naturwissenschaftliche Klasse*, vol. 5 (1928), p. 69; R. P. Boas, *Transactions of this Society*, vol. 40 (1936), p. 287. See also Van der Corput and Schaake, *Compositio Mathematica*, vol. 2 (1935), p. 321; R. J. Duffin and A. C. Schaeffer, *this Bulletin*, vol. 43 (1937), p. 554.

‡ S. Bernstein, *Communications de la Société Mathématique de Charkow*, (2), vol. 14. See also M. Riesz, *Acta Mathematica*, vol. 40 (1916), p. 337.

BERNSTEIN'S THEOREM. *Let $P(s)$ be a polynomial of degree n or less with real coefficients such that in the interval $(-1, 1)$ of the real axis $|P(s)| \leq 1$. If A and B are the semi-axes of an ellipse passing through the point s and having foci at the points 1 and -1 , then*

$$(9) \quad |P(s)| \leq (A + B)^n.$$

We shall prove that if $P(s)$ satisfies the conditions of Bernstein's theorem, then the stronger inequality

$$(10) \quad |P(s)| \leq \frac{1}{2} \{ (A + B)^n + (A + B)^{-n} \}$$

is satisfied.

PROOF. If $P(s)$ satisfies the conditions of Bernstein's theorem it is clear that $P(\cos z)$ is a polynomial of degree n in $\cos z$ and is bounded by 1 on the real axis, so it satisfies the conditions of Theorem 2 with $\lambda = n$. Then we have

$$(11) \quad |P(\cos z)| \leq \cosh ny,$$

where $z = x + iy$. Let $s = \sigma + it$ be any fixed point not in the interval $(-1, 1)$ of the real axis, and choose a z so that $s = \cos z$. Then we have the relations

$$\begin{aligned} \sigma &= \cos x \cosh y, \\ t &= -\sin x \sinh y, \end{aligned}$$

and on eliminating x we obtain the equation

$$\frac{\sigma^2}{A^2} + \frac{t^2}{B^2} = 1,$$

where

$$A = \cosh |y|, \quad B = \sinh |y|.$$

Thus the point s lies on an ellipse with center at the origin and semi-axes A and B , and the foci are at the points 1 and -1 since $A^2 - B^2 = 1$. Since $A + B = e^{|y|}$ we see that

$$\cosh ny = \frac{1}{2} \{ (A + B)^n + (A + B)^{-n} \},$$

and on putting this in (11) we have (10), namely,

$$|P(s)| = |P(\cos z)| \leq \frac{1}{2} \{ (A + B)^n + (A + B)^{-n} \}.$$

This is a "best possible" inequality in the sense that if $P(s)$ is the n th Tchebycheff polynomial, $T_n(s) = \cos(n \cos^{-1} s)$, then (10) becomes an equality along certain lines in the complex plane.